Shoaling of Solitary Waves

by

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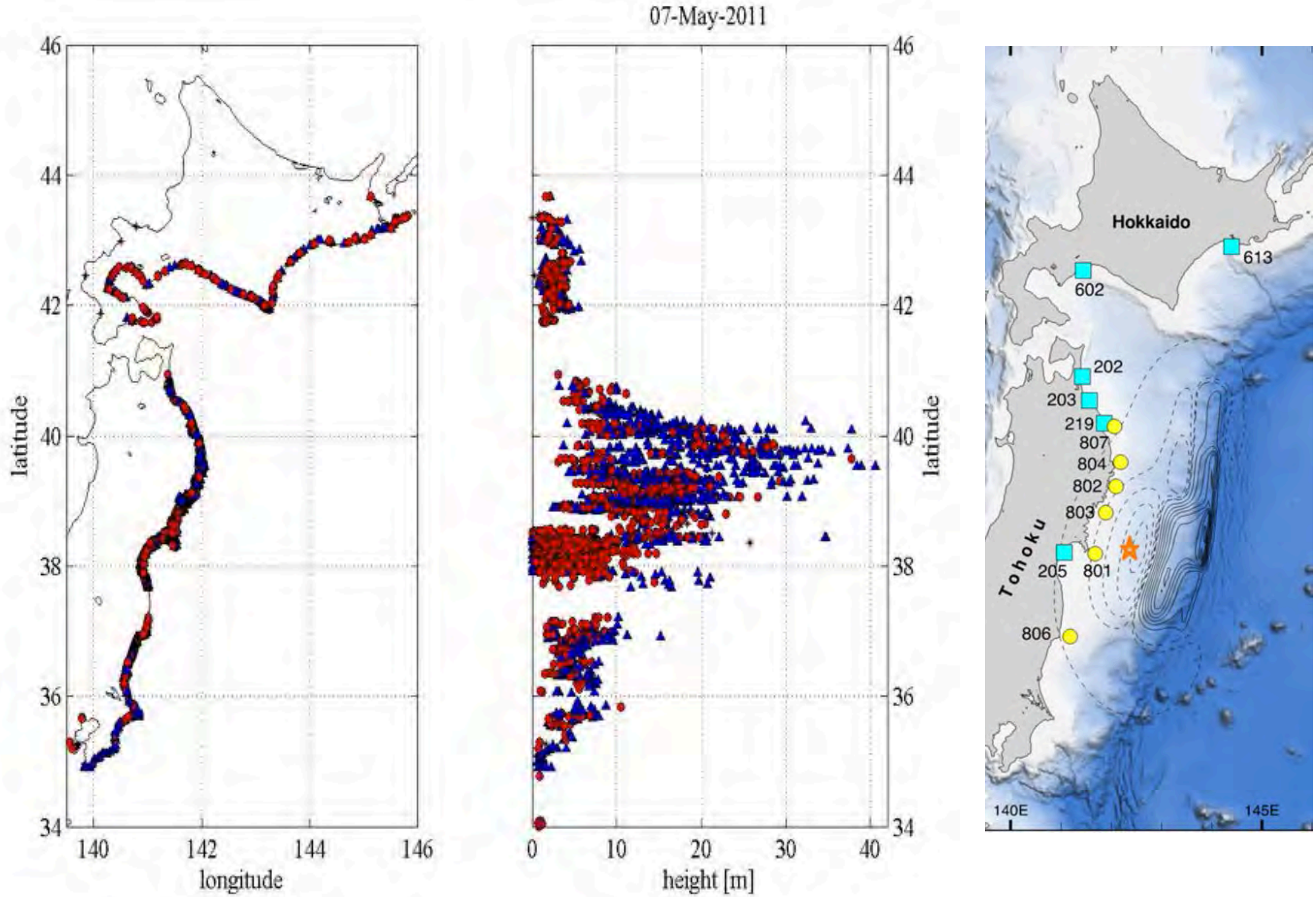
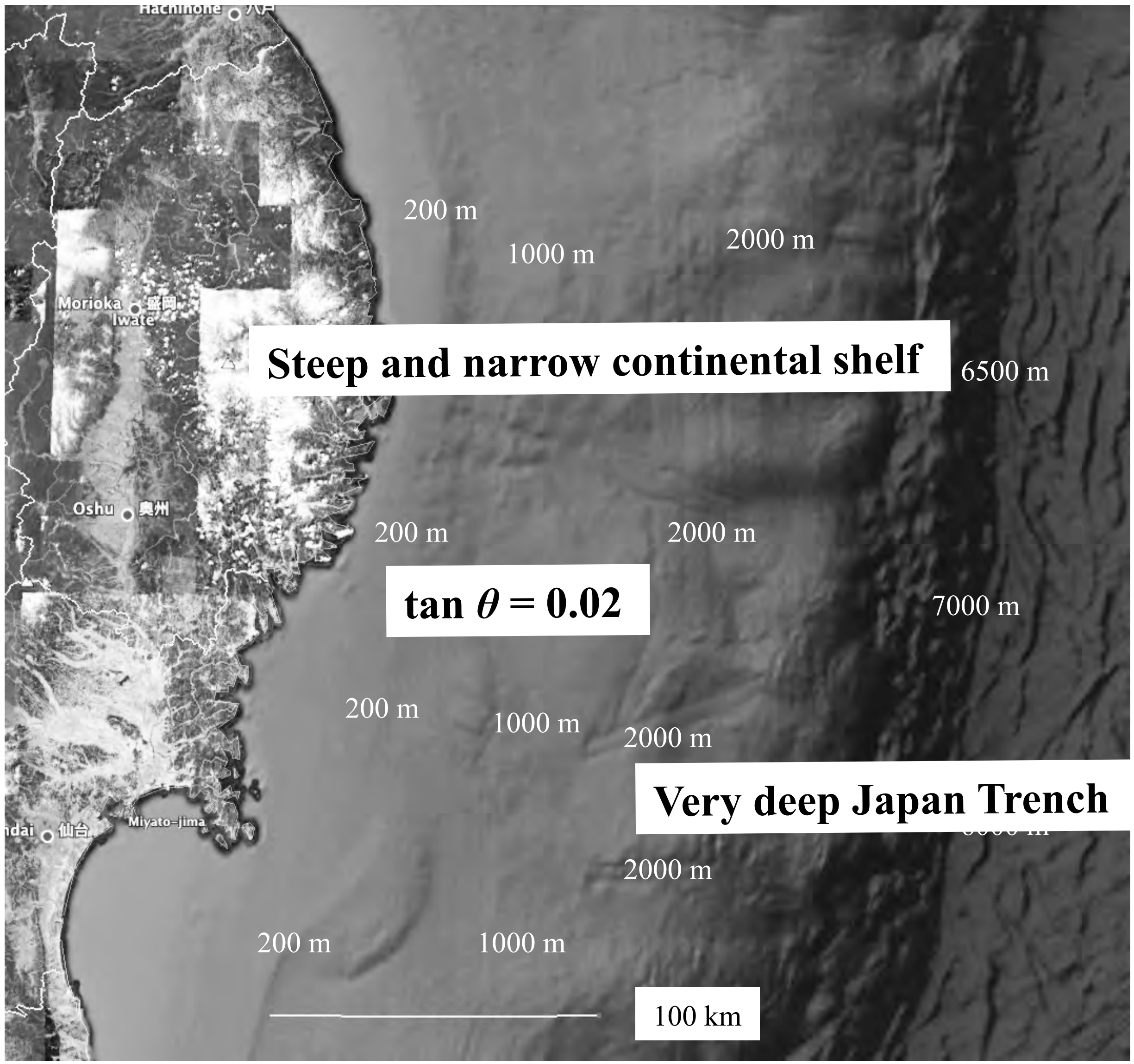
Water Waves, ICERM, Brown U., April 2017

Motivation

The 2011 Heisei Tsunami in Japan



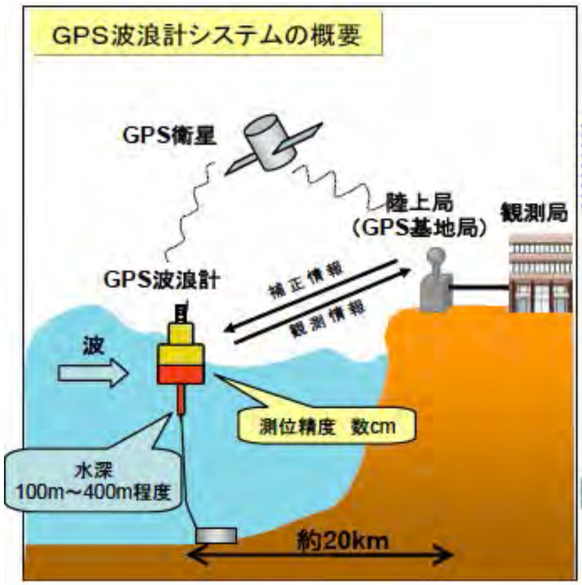
Bathymetry



Hachinohe
Morioka 盛岡
Iwate
Oshu 奥州
Miyato-jima
仙台
200 m
1000 m
2000 m
Steep and narrow continental shelf
6500 m
200 m
2000 m
tan θ = 0.02
7000 m
200 m
1000 m
2000 m
Very deep Japan Trench
2000 m
200 m
1000 m
100 km

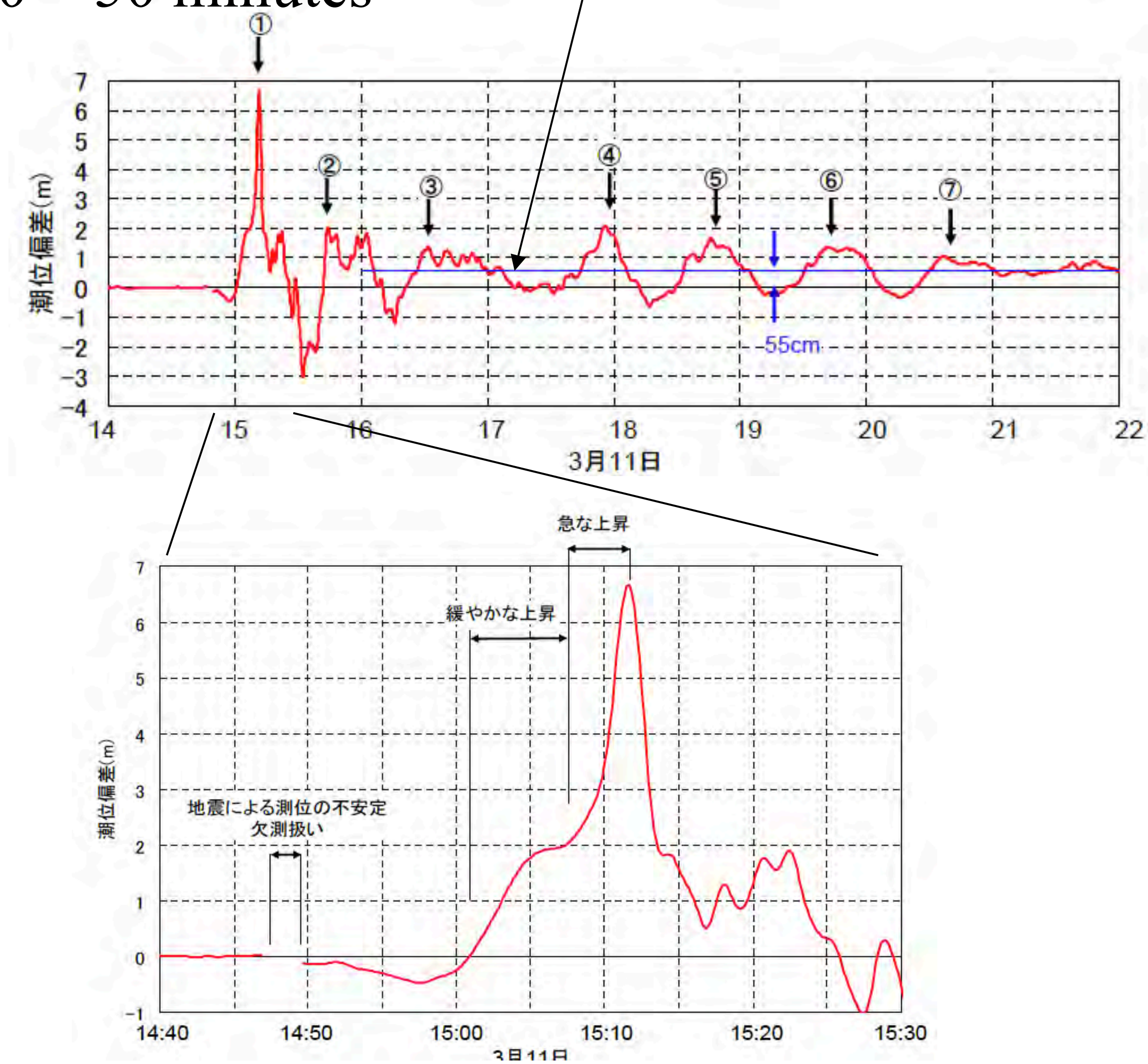
GPS Wave Gage

Water depth 204 m

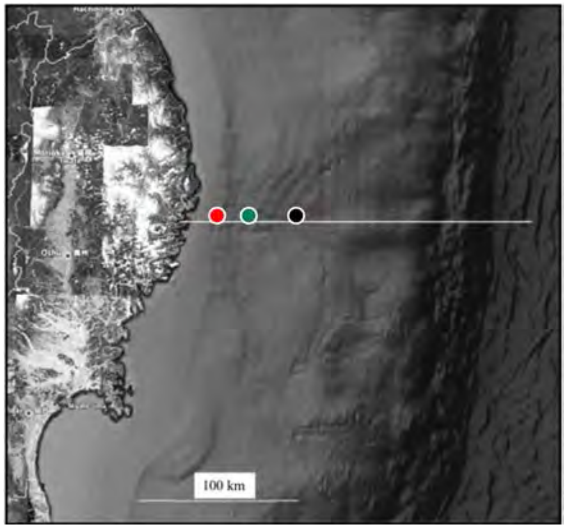


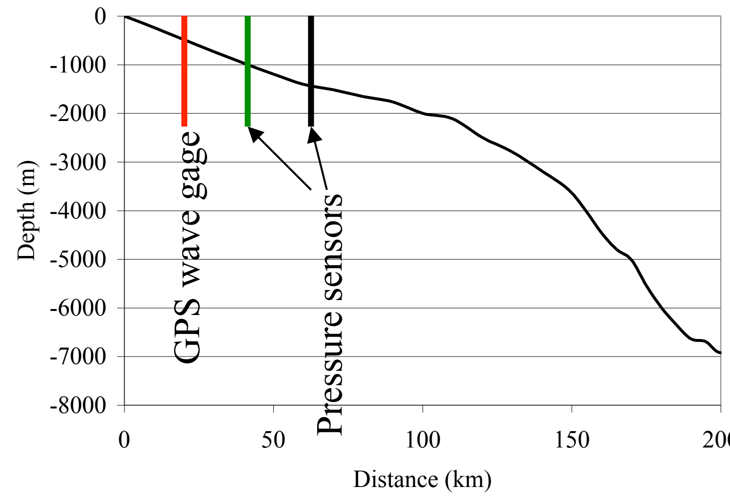
Wave period
40 ~ 50 minutes

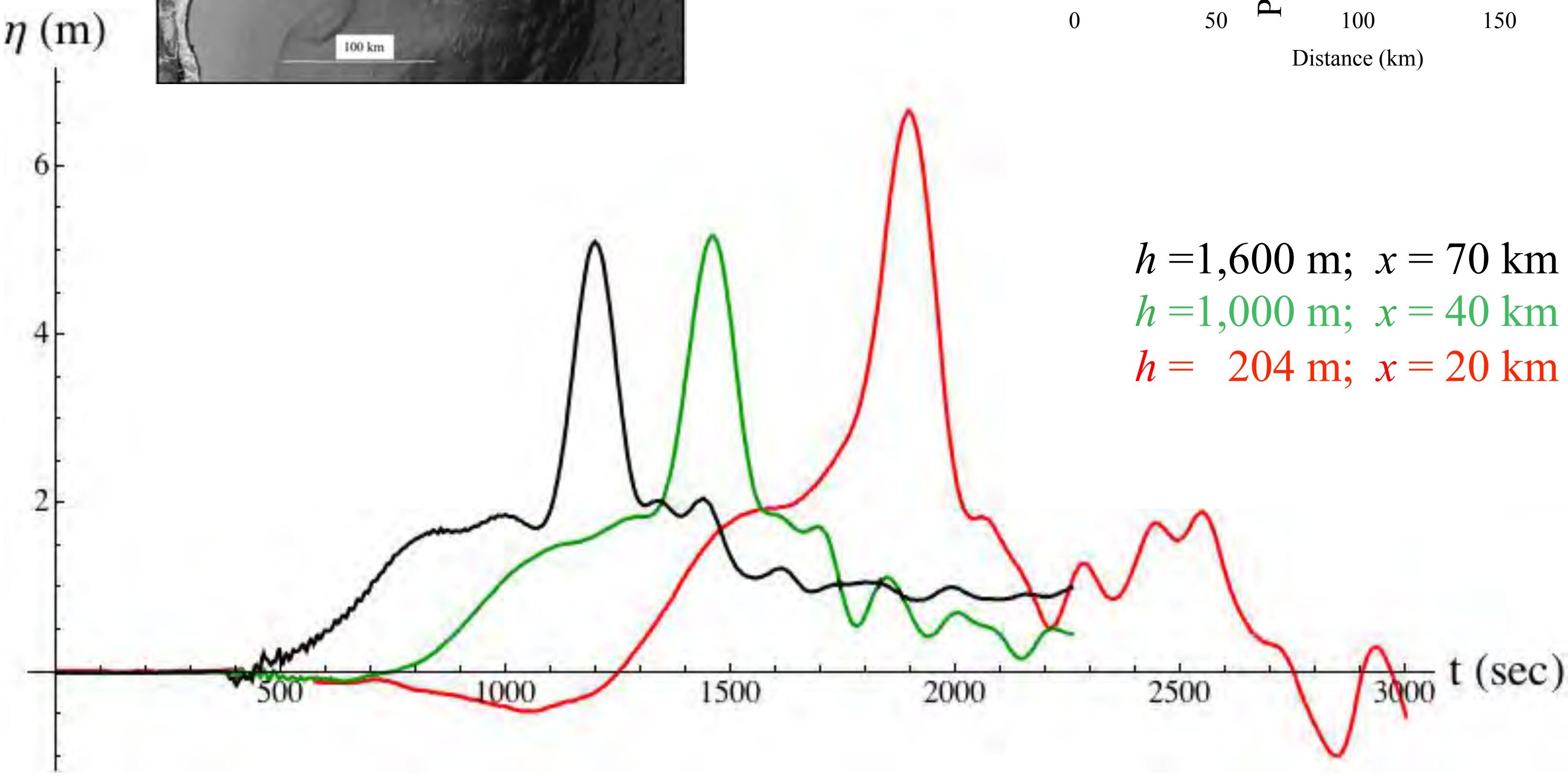
55 cm land subsidence



Seabed Pressure Data and GPS Wave Gage Off Kamaishi







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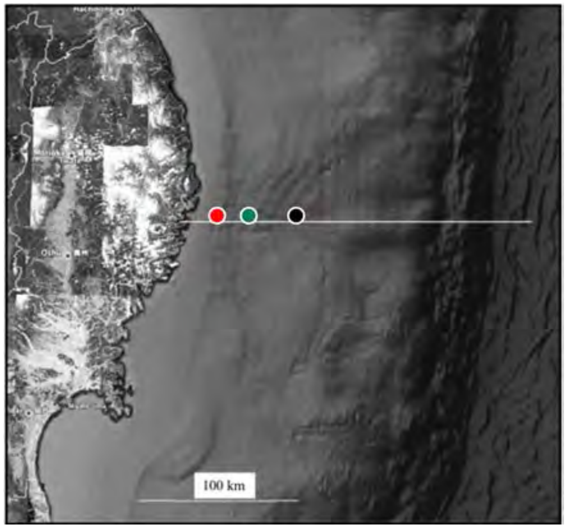
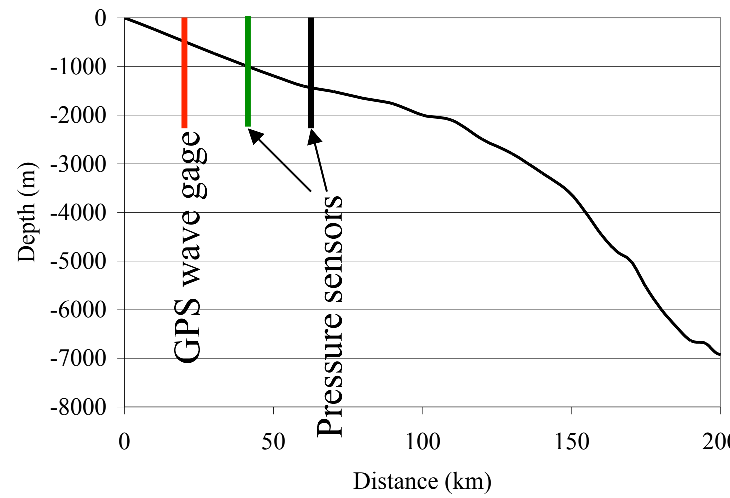
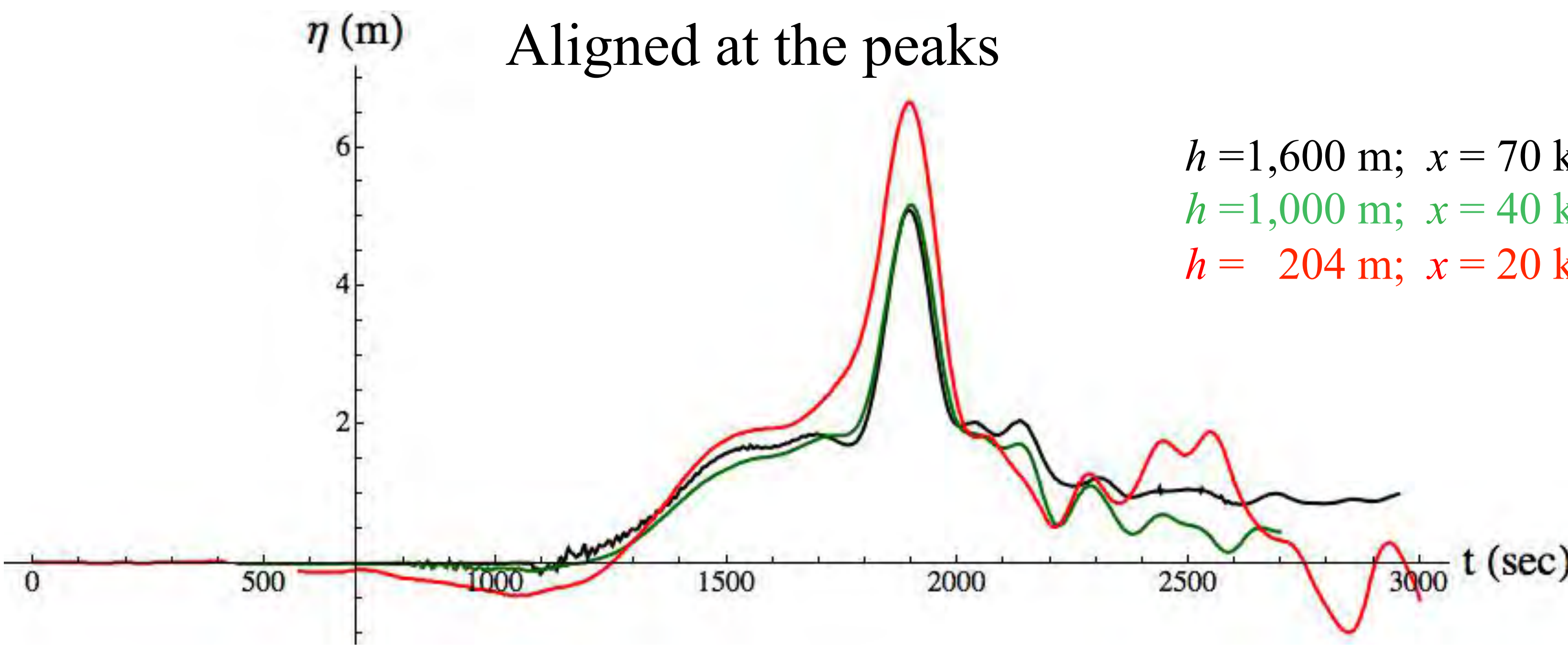
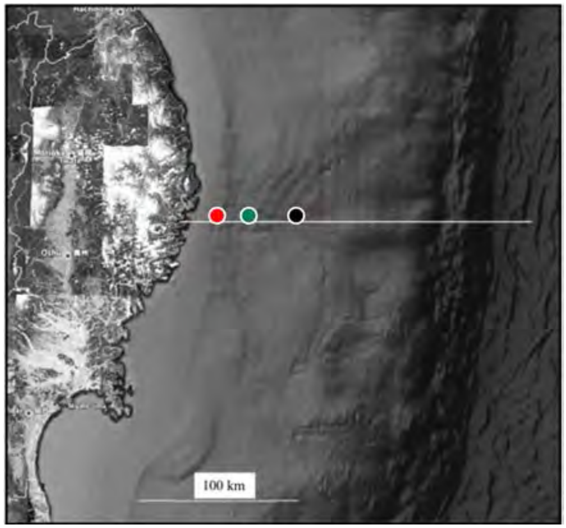


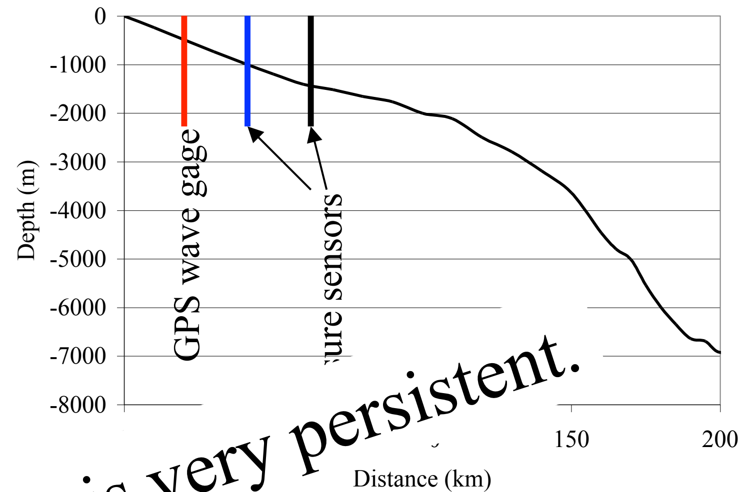


Aligned at the peaks

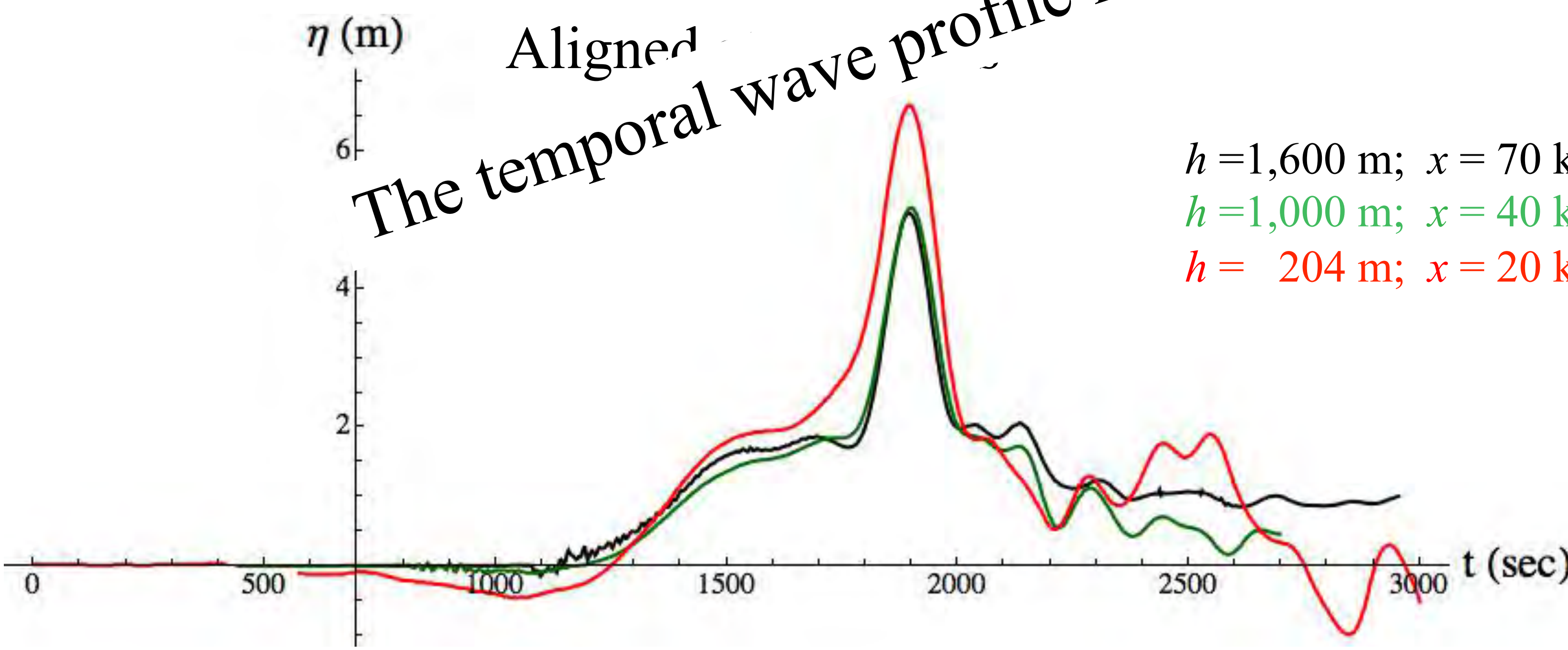


Seabed Pressure Data and GPS Wave Gage Off Kamaishi





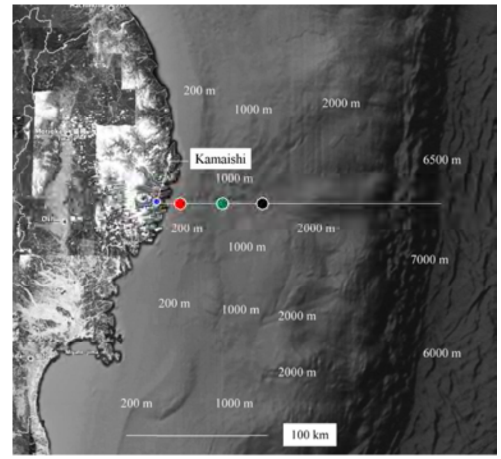
The temporal wave profile is very persistent.

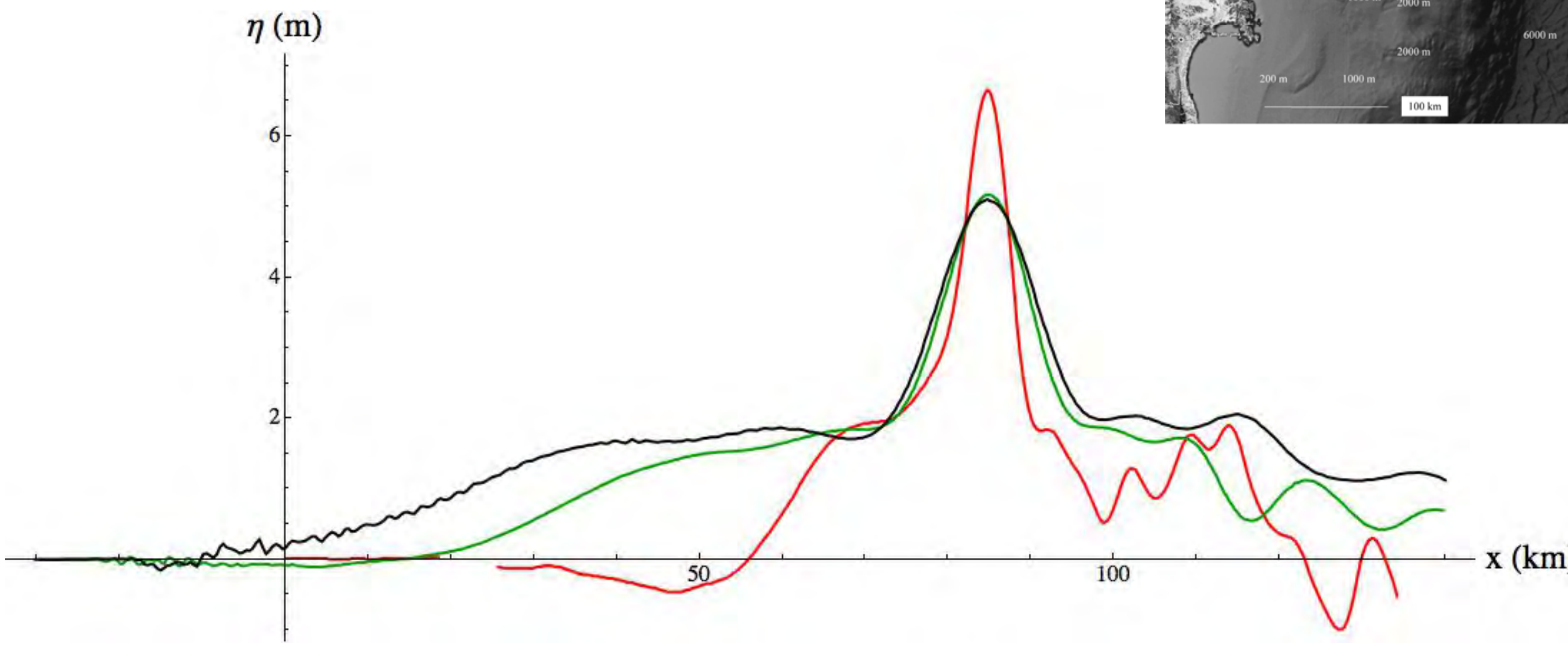


Spatial Profiles

The sharply peaked wave riding on the broad tsunami base appears to maintain its “symmetrical” waveform with increase in amplitude and narrow in wave breadth.

Simple conversion ($x = c\ t$) shows that the length of the peaky wave is ~ 25 km: not too long.





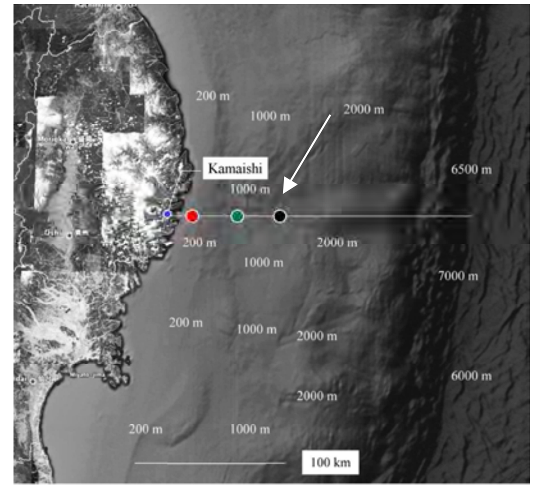
Can this tsunami be considered as a soliton?

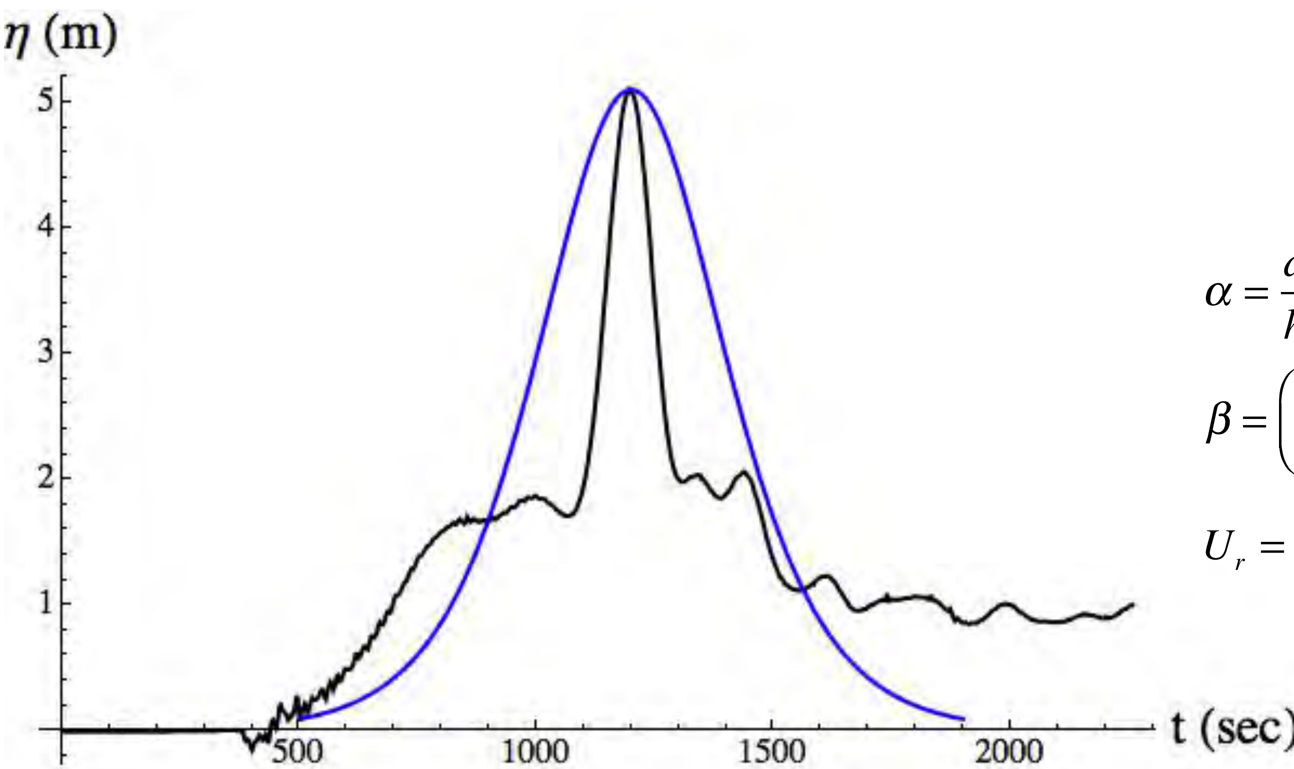
Seabed Pressure Transducers (ERI, University of Tokyo)

h =1,600 m; x = 70 km.

$$\eta = a\ \mathrm{sech}^2\left[\sqrt{\frac{3a}{4h^3}}\left(x - c_0\left(1 + \tfrac{a}{2h}\right)t\right)\right]$$

The breadth of the wave profile 2λ is taken at $\eta = 0.51\ a$. With this choice of length scale, the Ursell number of a solitary wave is $U_r = \alpha/\beta = 1.0$, where $\alpha = a/h$; $\beta = (h/\lambda)^2$.





$$\alpha = \frac{a}{h} = \frac{5.1}{1600} \approx 0.0032$$

$$\beta = \left(\frac{h}{\lambda}\right)^2 = \left(\frac{1600}{9100}\right)^2 \approx 0.031$$

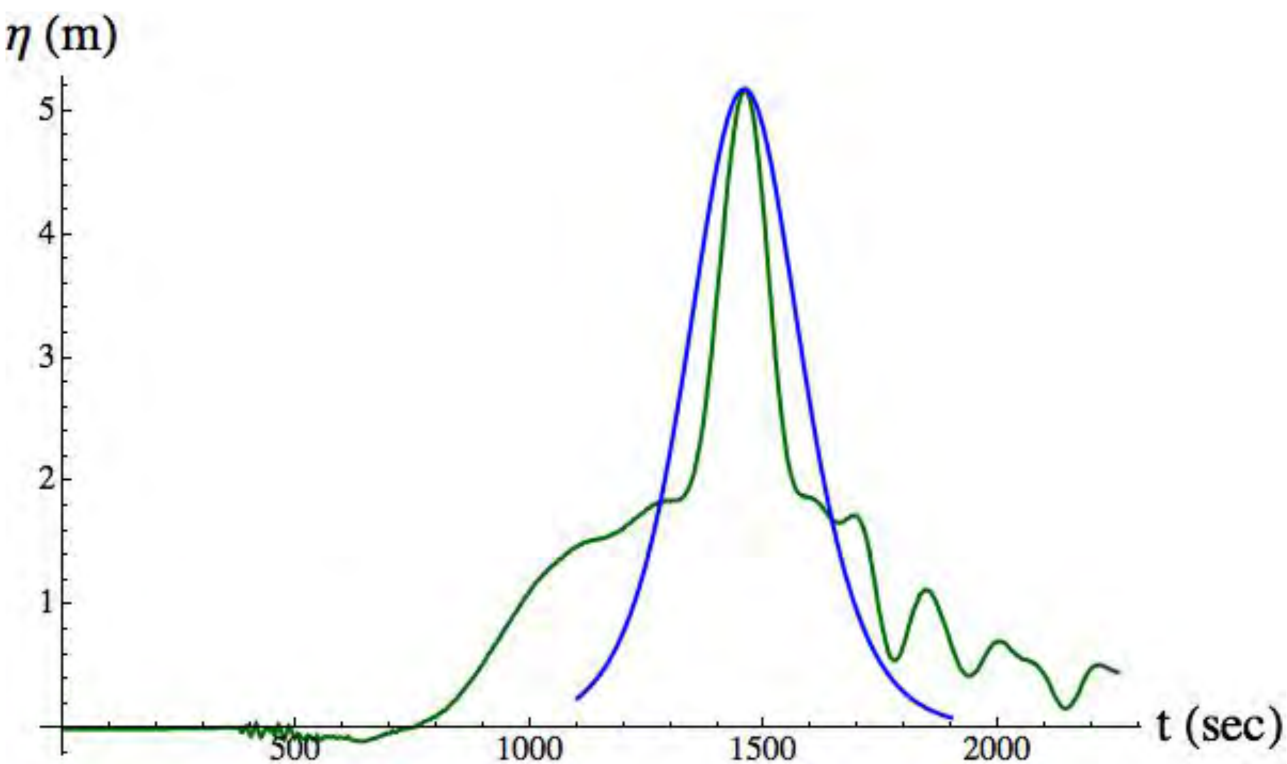
$$U_r = \frac{\alpha}{\beta} = 0.10 \quad \text{(The Ursell Number)}$$

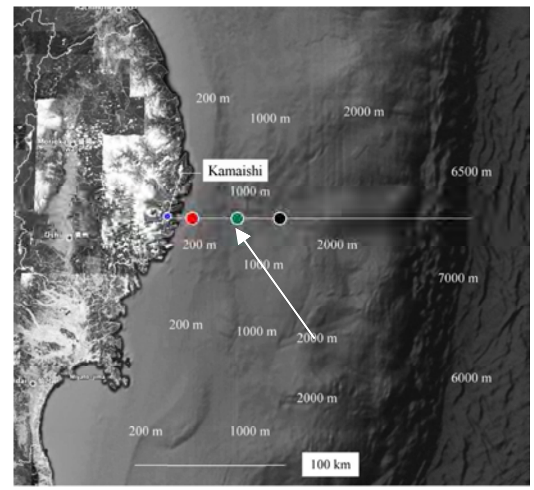
Seabed Pressure Transducers (ERI, University of Tokyo)

h =1,000 m; x = 40 km.

The wave form becomes closer to that of soliton.

$$\eta = a\ \mathrm{sech}^2\left[\sqrt{\frac{3a}{4h^3}}\left(x - c_0\left(1+\tfrac{a}{2h}\right)t\right)\right]$$





$$\alpha = \frac{a}{h} = \frac{5.2}{1000} \approx 0.0052$$

$$\beta = \left(\frac{h}{\lambda}\right)^2 = \left(\frac{1000}{7900}\right)^2 \approx 0.016$$

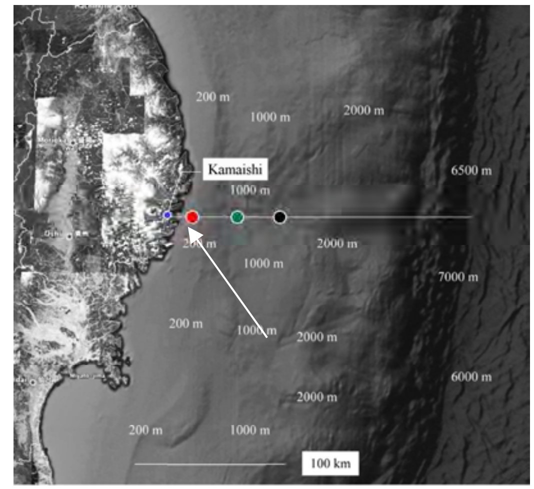
$$\frac{\alpha}{\beta} = 0.33$$

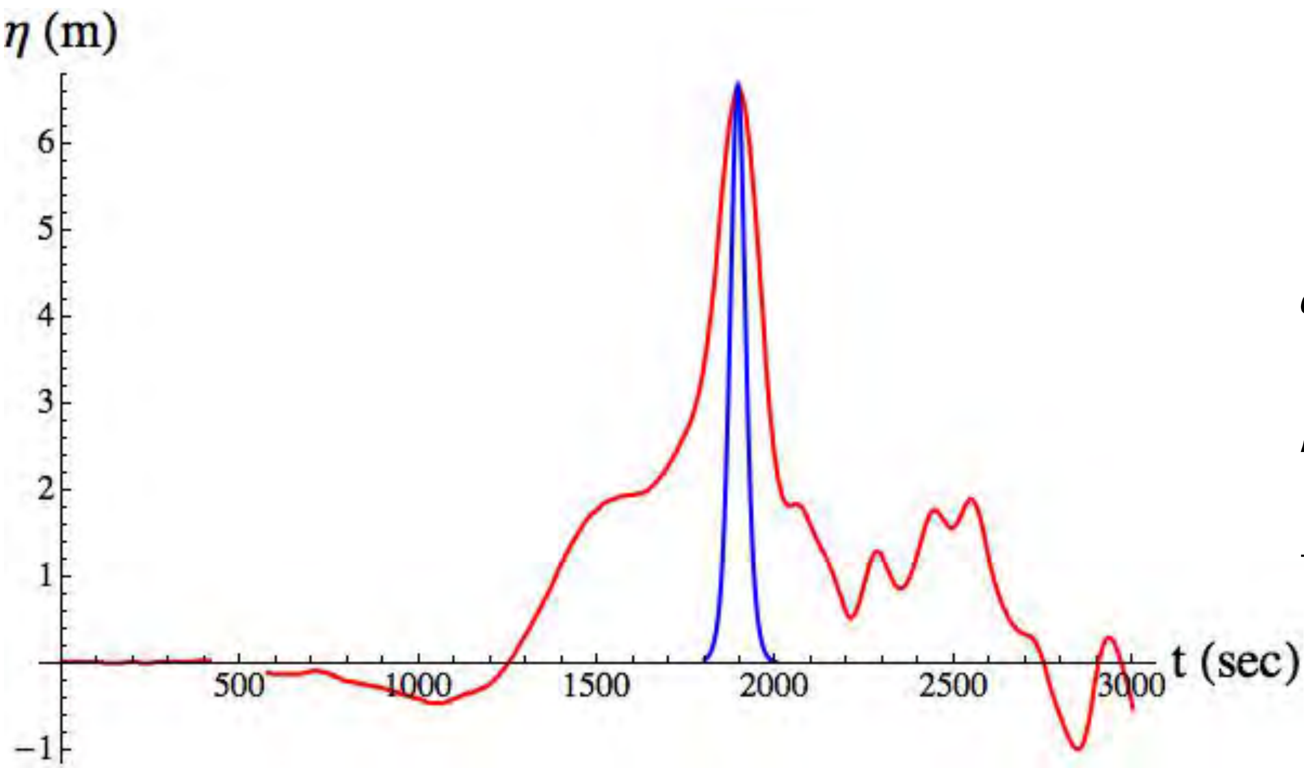
GPS Wave Gage: 20 km off Kamaishi

$h = 204\text{ m};\ \ x = 20\text{ km}$

The Spike Riding on the Broad Tsunami resembles a soliton profile?

$$\eta = a\ \mathrm{sech}^2\left[\sqrt{\frac{3a}{4h^3}}\left(x - c_0\left(1 + \tfrac{a}{2h}\right)t\right)\right]$$

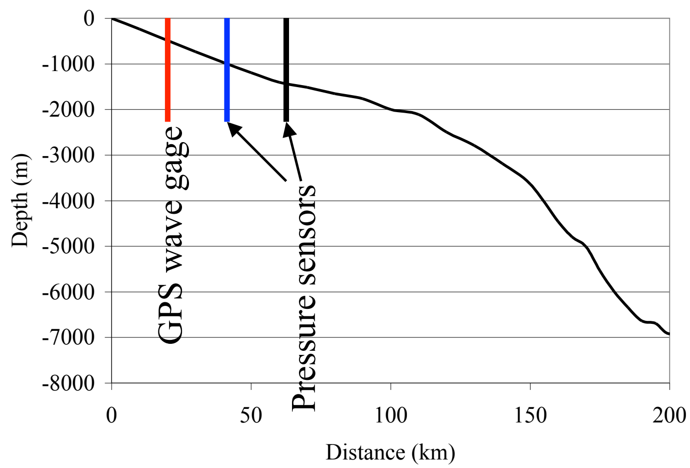


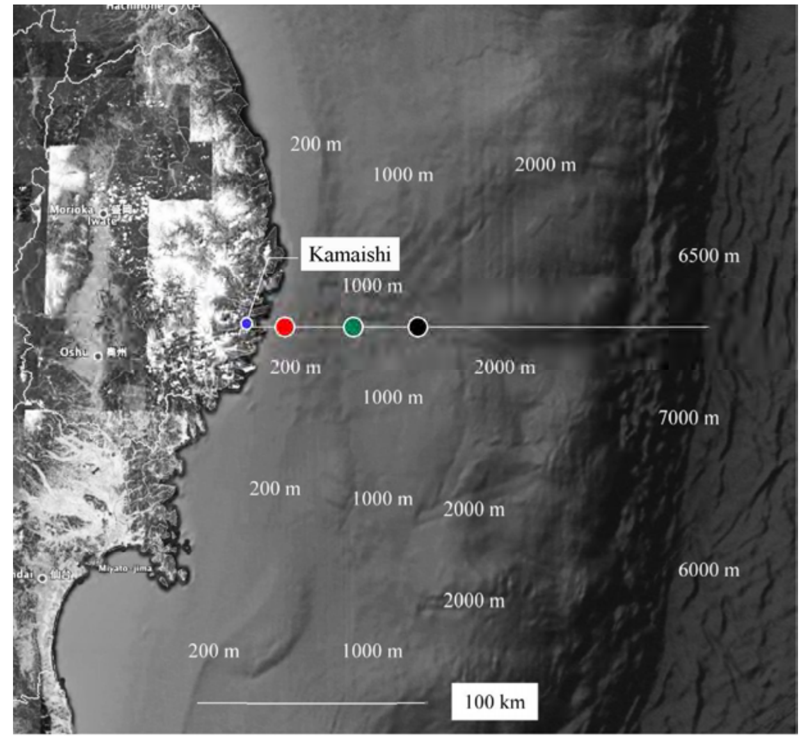


$$\alpha = \frac{a}{h} = \frac{6.7}{204} \approx 0.033$$

$$\beta = \left(\frac{h}{\lambda}\right)^2 = \left(\frac{204}{4000}\right)^2 \approx 0.0026$$

$$\frac{\alpha}{\beta} = 12.7$$





Tsunami parameters: nonlinearity *a*

Frequency dispersion β

Ursell number U_r

Seafloor slope *q*.

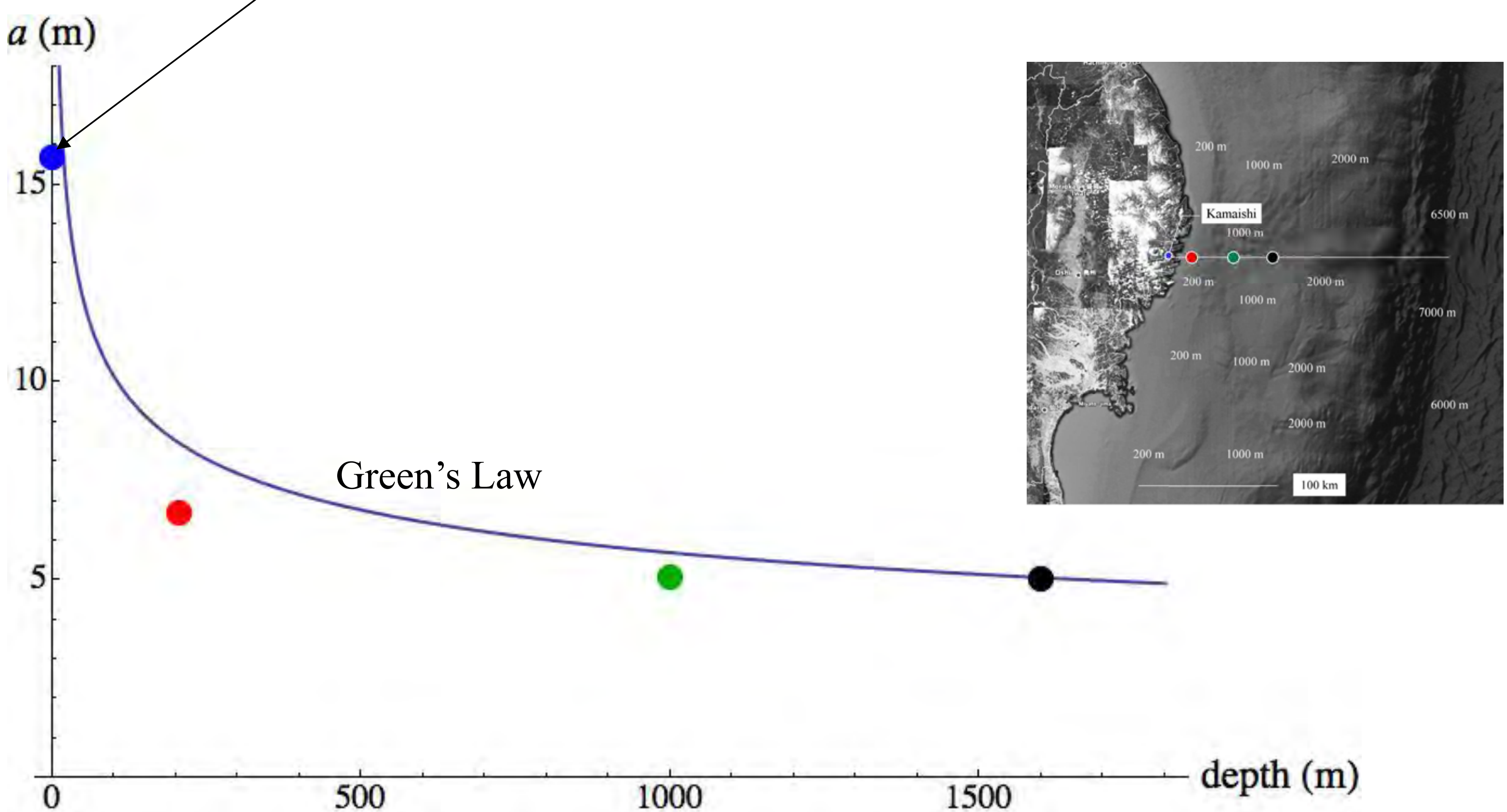
	$\alpha = a/h$	$\beta = (h/\lambda)^2$	$U_r = \alpha/\beta$	θ
TM1	0.0032	0.031	0.1	*
TM2	0.0052	0.016	0.3	0.021
GPS	0.033	0.0026	13.0	0.022

It is more or less a linear long wave with a finite seabed slope.

Tsunami amplification (Shoaling)

Green's Law: $a \propto h^{-1/4}$: (based on linear shallow-water-wave theory)

Measured runup heights onshore near Kamaishi: 15.7 m ± 6.7 m.



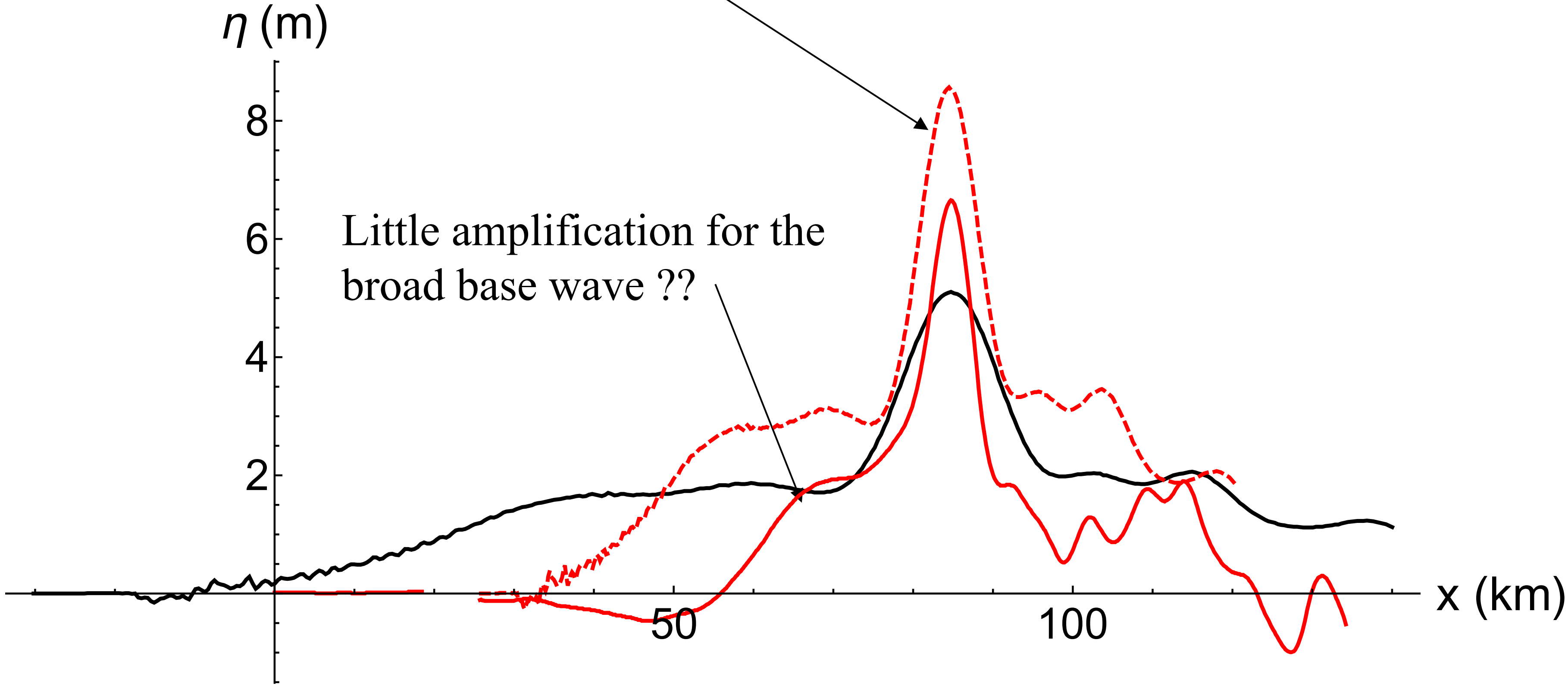
Does Green's law work?

$$a \propto h^{-1/4}$$

Predicted waveform at x = 20 km using Green's law from the data at x = 70 km.

h =1,600 m; x = 70 km
h = 204 m; x = 20 km

Little amplification for the broad base wave ??



What We Observed from the Field Data

Wave data along the east-to-west transect from Kamaishi.

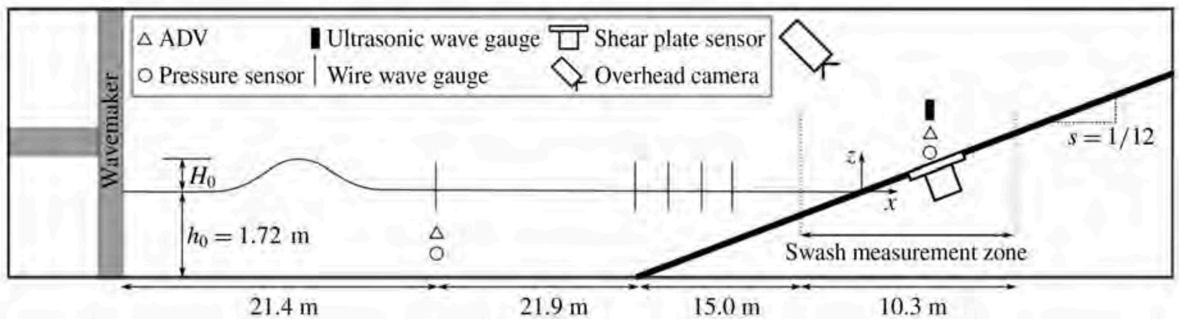
- A unique tsunami waveform did not change much from the offshore location to the nearshore location: the waveform is comprised of a narrow spiky wave riding on the broad tsunami base at its rear portion.

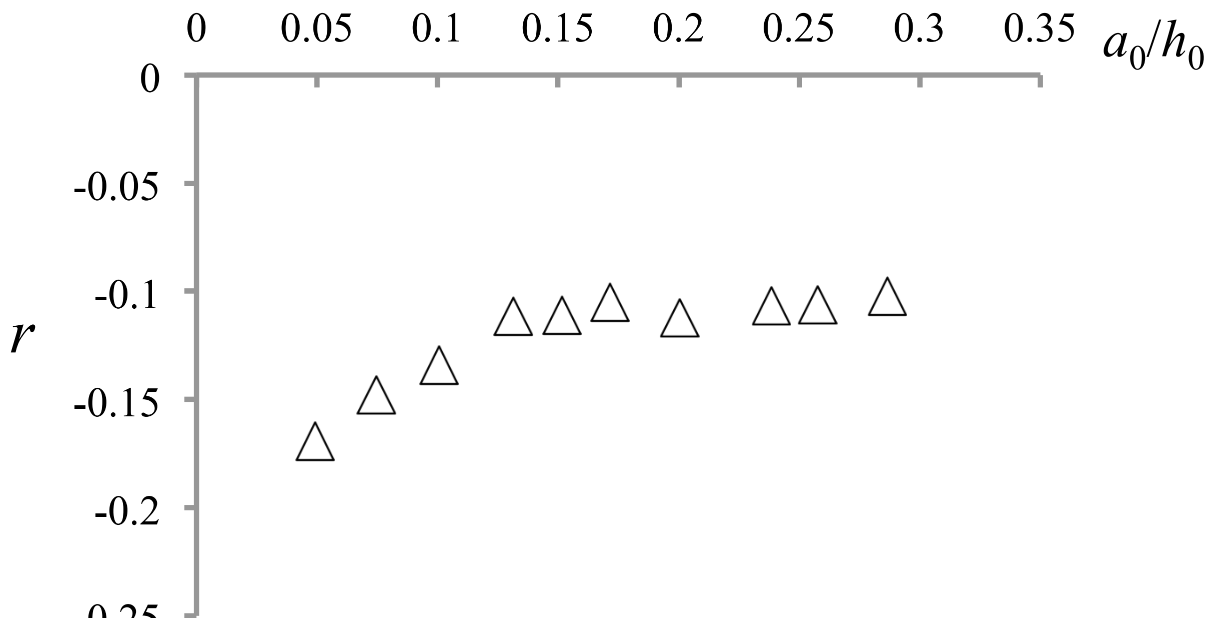
- In spite of the persistent symmetrical waveform, the tsunami evolution is quite different from that of a soliton – it is not the adiabatic evolution.

- As the tsunami approaches the shore, there is practically no amplification of the broad base portion of the tsunami, although the amplitude of the narrow spiky tsunami riding on the broad portion increased but not as fast as the prediction of Green's law.

Laboratory Data by Pujara, Liu, & Yeh 2015

Does Green's law work: $r = -\frac{1}{4}$?





$$a \propto h^r$$

$$|r| < 1/4$$

Solitary Wave Shoaling in the Laboratory?

- Laboratory data show that shoaling amplification of the solitary waves is slower than that of Green's law. As the nonlinearity increases, $\rightarrow \; a \propto h^{-1/10}$
- This is a consistent trend with the field observation.

Background

Grimshaw (1970, 1971); Johnson (1973);

Adiabatic: the depth variation occurs on a scale that is slower than the evolution scale of the wave, so that the wave deforms but maintains its identity of soliton.

Dimensionally, the adiabatic solution can be inferred:

$$\eta = a_0 \operatorname{sech}^2\left[\sqrt{\frac{3a_0}{4h_0^3}}\left(x - c_0\left(1+\tfrac{a}{2h}\right)t\right)\right]$$

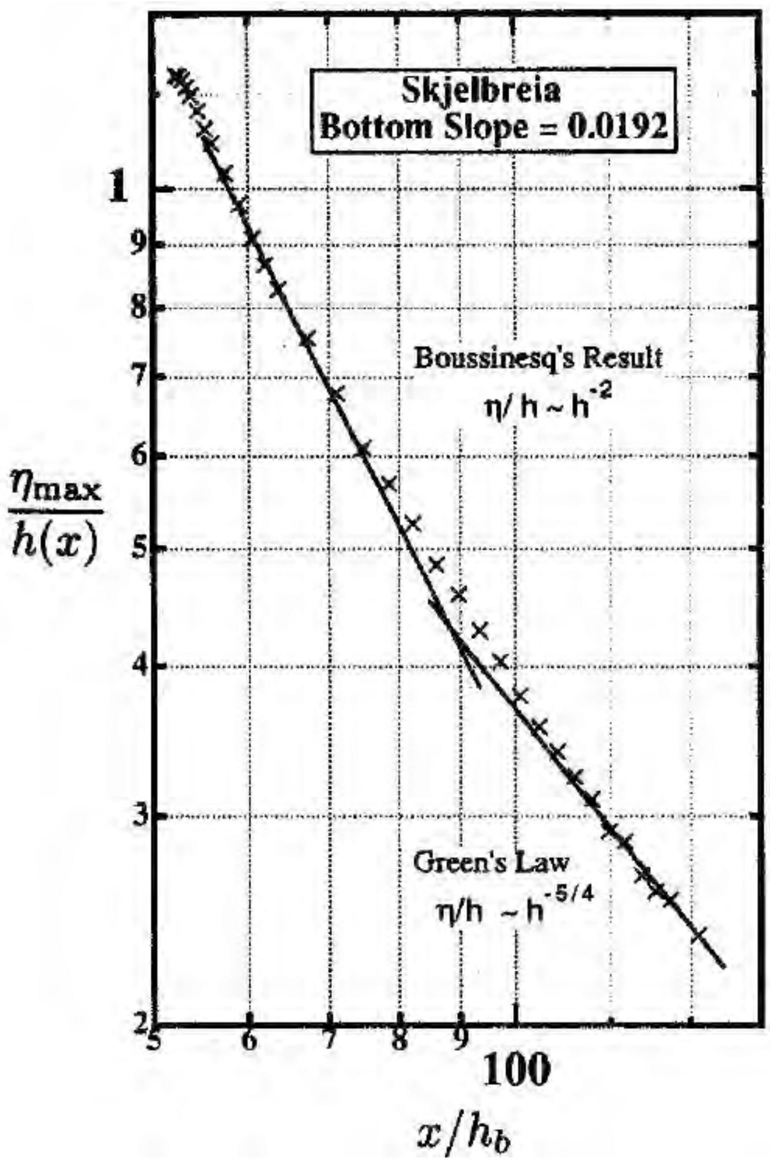
$$\Downarrow$$

$$\eta = \frac{a_0 h_0}{h} \operatorname{sech}^2\left[\sqrt{\frac{3a_0}{4h}}\,\frac{1}{h}\left(x - ct\right)\right]; \quad c = c(a,h)$$

$$ah = a_0 h_0: \quad a \propto h^{-1}$$

This can be formally shown with the conservation of wave action flux.

Synolakis and Skjelbreia (1993)



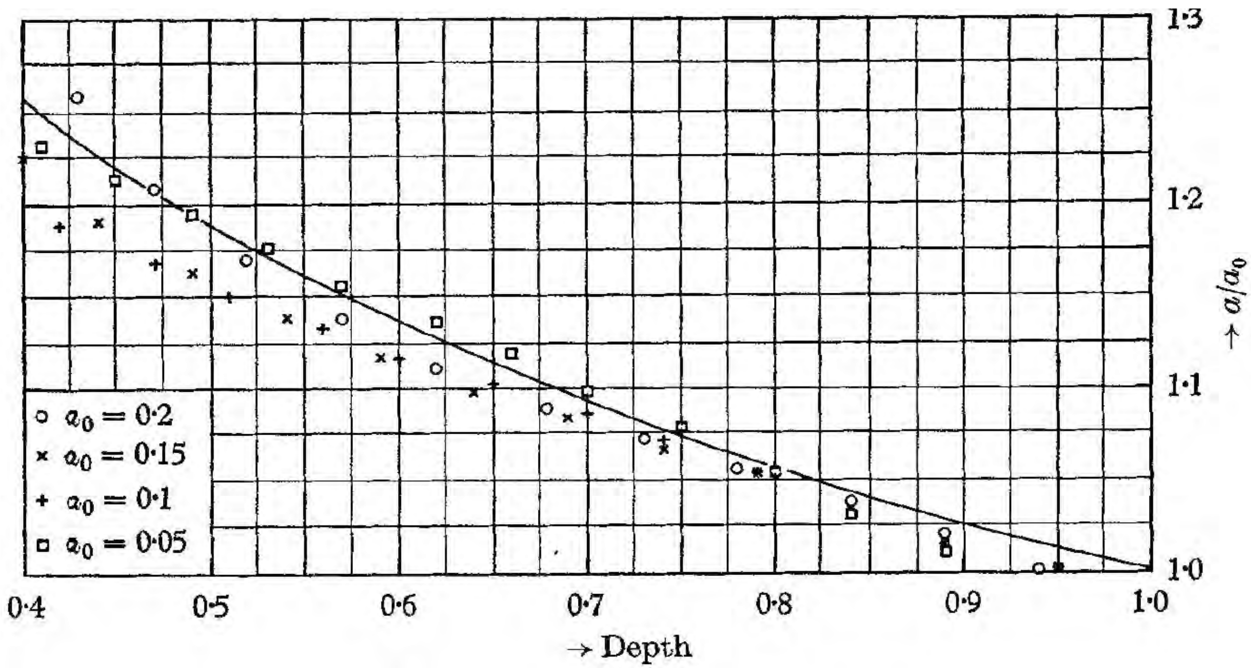
- Exact solution to the "linear non-dispersive" shallow-water wave equation with a solitary-wave initial condition yields Green's law in the offshore region (Synolakis, 1991)

- Laboratory Observation: Two zones of gradual shoaling and rapid shoaling: a) Green's law, b) adiabatic.

Peregrine (1967)

$$\begin{cases} \bar{u}_t + \bar{u}\,\bar{u}_x + \eta_x = \tfrac{1}{3}\theta^2 x^2 \bar{u}_{xxt} + \theta^2 x \bar{u}_{xt}, \\ \eta_t + \left[(\theta x + \eta)\bar{u}\right]_x = 0. \end{cases}$$

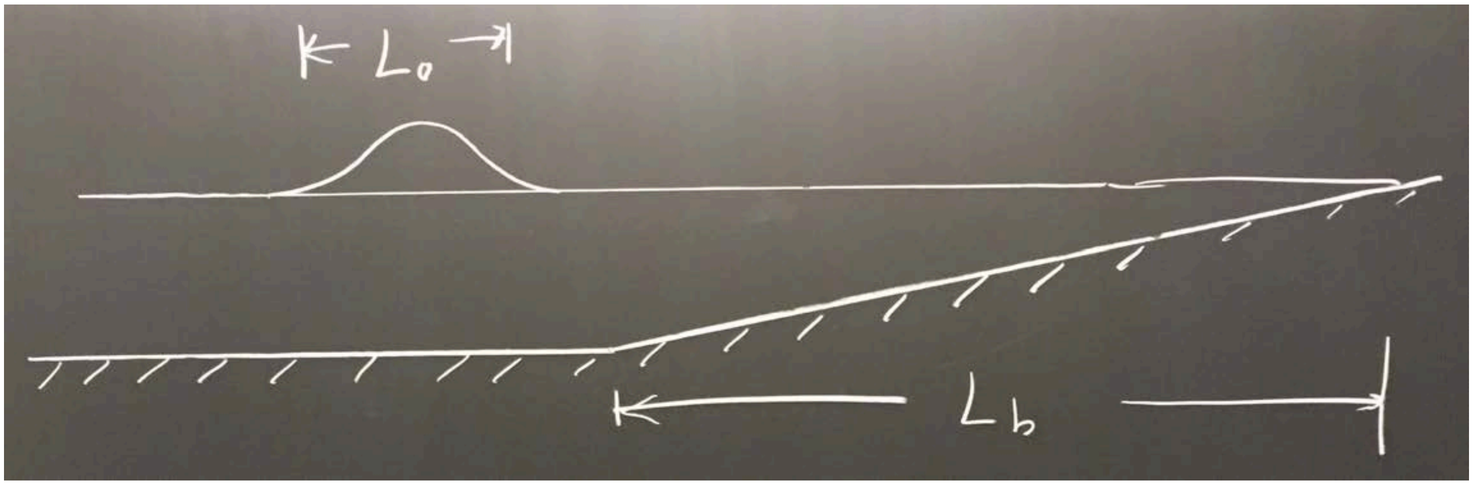
Extension of the Boussinesq equation:
x points offshore from the initial shoreline



Numerical results of the solitary-wave shoaling: tan $\theta = 1/20$. The solid line represents Green's law.

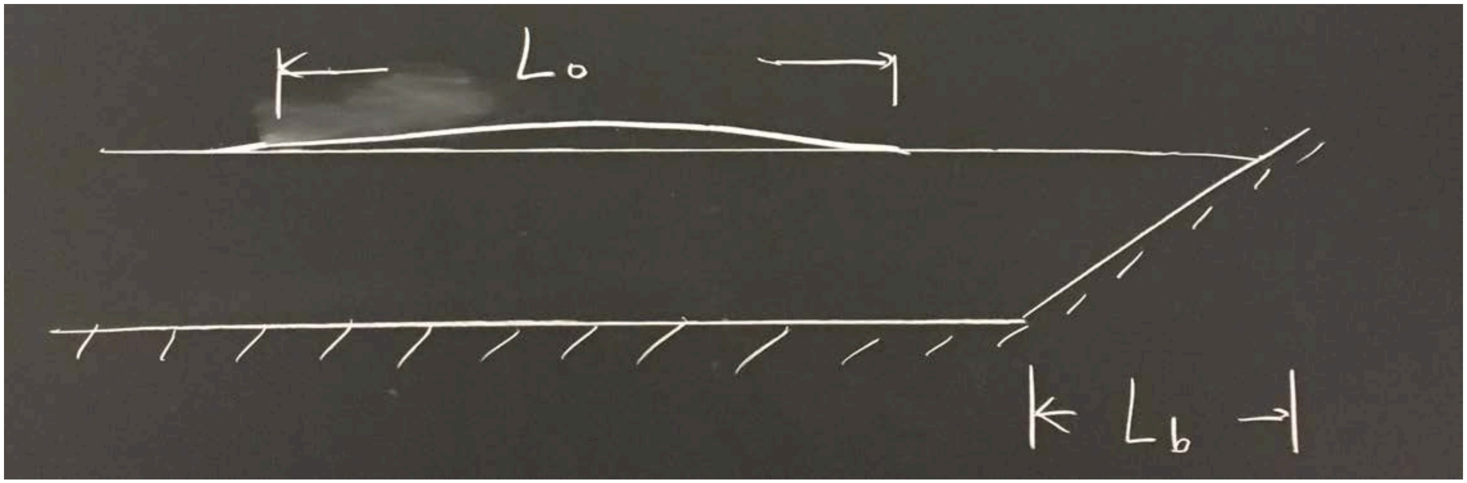
Preliminary Considerations

Preliminary Considerations



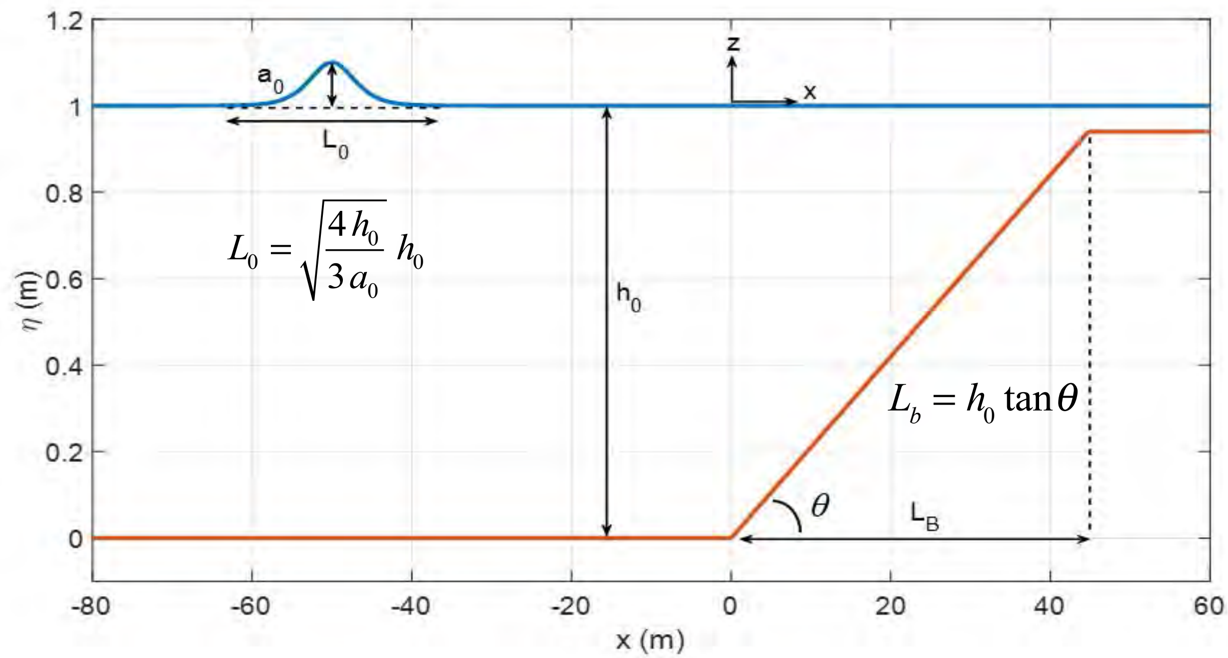
- When the beach slope is mild and the wave amplitude is large, i.e. L_b large and L_0 small, then, it is reasonable to anticipate for the adiabatic evolution process.
- A problem is that the incident wave can break in the early stage of the shoaling process, because of the finite initial amplitude.

Preliminary Considerations



- When the beach slope is steep and the wave amplitude is small, i.e. L_b small and L_0 large, then, the wave as a whole may not have chance to shoal due to the short shoaling distance.
- The wave length be too long so that only a portion of the waveform be influenced by the sloping bed. For this situation, we anticipate little shoaling of the incident wave, but the wave may amplify due to reflection.

Preliminary Considerations



$$\alpha_0 = \frac{a_0}{h_0}$$

- It is important is recognize that, once we deal with a sloping bed, the propagation domain is no longer infinite, but finite. The steeper the slope, the shorter the available propagation distance.

- $\gamma = L_0/L_b$ must be a relevant parameter to characterize the solitary wave shoaling.

$$\boxed{\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}}$$

Analytical Considerations

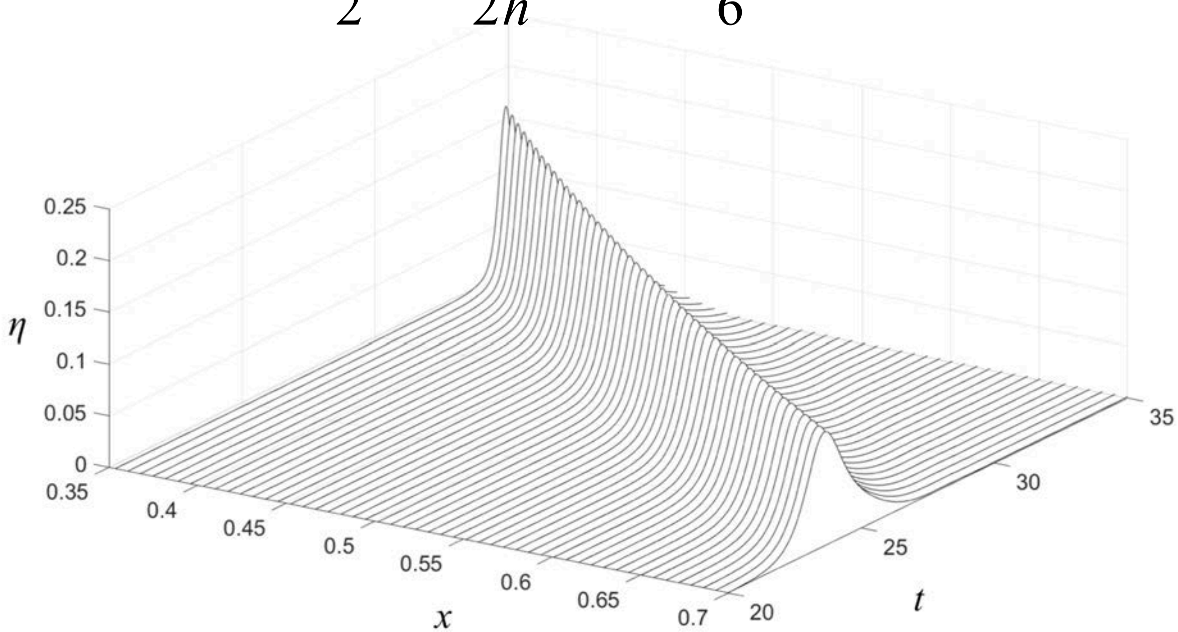
Variable Coefficient Korteweg-de Vries Equation

The vKdV equation: $$\eta_t + c\,\eta_x + \frac{c_x}{2}\eta + \frac{3c}{2h}\eta\eta_x + \frac{c\,h^2}{6}\eta_{xxx} = 0$$

Here $\eta = \eta(x,t)$ and $c(x) = \sqrt{g\,h(x)}$, in which $h(x) = x\tan\theta + h_{0.}$

The extremum of $\eta\,(x)$ happens when $\partial_t\,\eta = 0$. Hence the following equation must satisfy for the envelope of η:

$$c\,\eta_x + \frac{c_x}{2}\eta + \frac{3c}{2h}\eta\eta_x + \frac{c\,h^2}{6}\eta_{xxx} = 0$$



Variable Coefficient Korteweg-de Vries Equation

For the amplitude envelope, $\partial_t \eta = 0$.

$$c\,\eta_x + \frac{c_x}{2}\eta + \frac{3c}{2h}\eta\,\eta_x + \frac{c\,h^2}{6}\eta_{xxx} = 0$$

After normalizing the variables ($\zeta = \eta(h(x))/a_0$, $h = h/h_0$), we can write:

$$h\zeta' + \frac{1}{4}\zeta + \frac{3}{2}\alpha_0\,\zeta\,\zeta' + \frac{1}{6}h^3\tan^2\theta\,\zeta''' = 0$$

where $\alpha_0 = a_0/h_0$.

Linear Non-Dispersive Case

$$\boxed{h\zeta' + \frac{1}{4}\zeta = 0}$$

Therefore $\zeta = C_0\,h^{-1/4}$. This is Green's law for linear monochromatic waves.

Nonlinear Non-Dispersive Case

$$h\zeta' + \frac{1}{4}\zeta + \frac{3}{2}\alpha_0 \zeta \zeta' + \frac{1}{6} h^3 \tan^2\theta\, \zeta''' = 0$$ becomes $$\boxed{h\zeta' + \frac{1}{4}\zeta + \frac{3}{2}\alpha_0 \zeta \zeta' = 0}$$

θ dependency is dropped.

This can be arranged as: $\zeta' = \dfrac{-1/4}{(\zeta/h)^{-1} + \frac{3}{2}\alpha_0}$

Taking $\zeta = h\upsilon(h)$ so that $\zeta' = h\upsilon' + \upsilon$ yields: $\dfrac{1 + \frac{3}{2}\alpha_0 \upsilon}{\upsilon(\frac{5}{4} + \frac{3}{2}\alpha_0 \upsilon)}\, d\upsilon = -\dfrac{1}{h}\, dh$

Integration yields: $\frac{4}{5}\ln \upsilon + \frac{1}{5}\ln\left(\alpha_0 \upsilon + \frac{5}{6}\right) = -\ln h + \text{constant}$

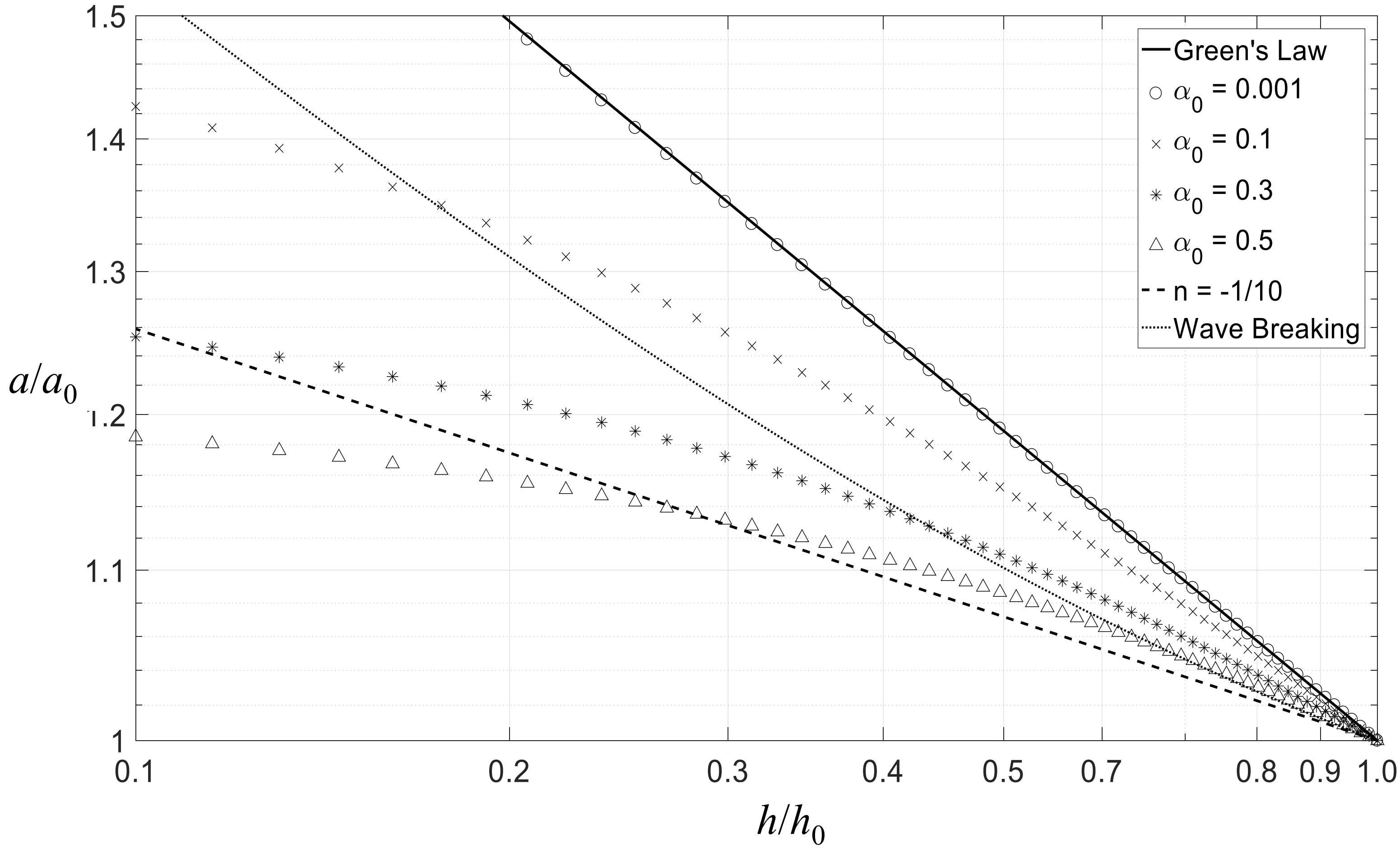
Therefore, $\left(\dfrac{\zeta}{h}\right)^{4/5} \left(\alpha_0 \dfrac{\zeta}{h} + \dfrac{5}{6}\right)^{1/5} = C_0\, h^{-1}$

Note that this reduces to Green's law for $\alpha_0 << 1$.

This equation can be written as $\alpha_0 \zeta^5 + \frac{5}{6} h \zeta^4 - C_0^5 = 0$

There are five roots, two of which are complex, another two which are negative, and one that is positive. It must therefore be that the positive real root represents the physical amplitude.

Nonlinear Non-Dispersive Case



$$\alpha_0 = \frac{a_0}{h_0}$$

The wave breaking criterion, $a/h = 0.78$, is used here.

The solution is independent of the beach slope.

$$a \propto h^{-r};\ r < \tfrac{1}{4}$$

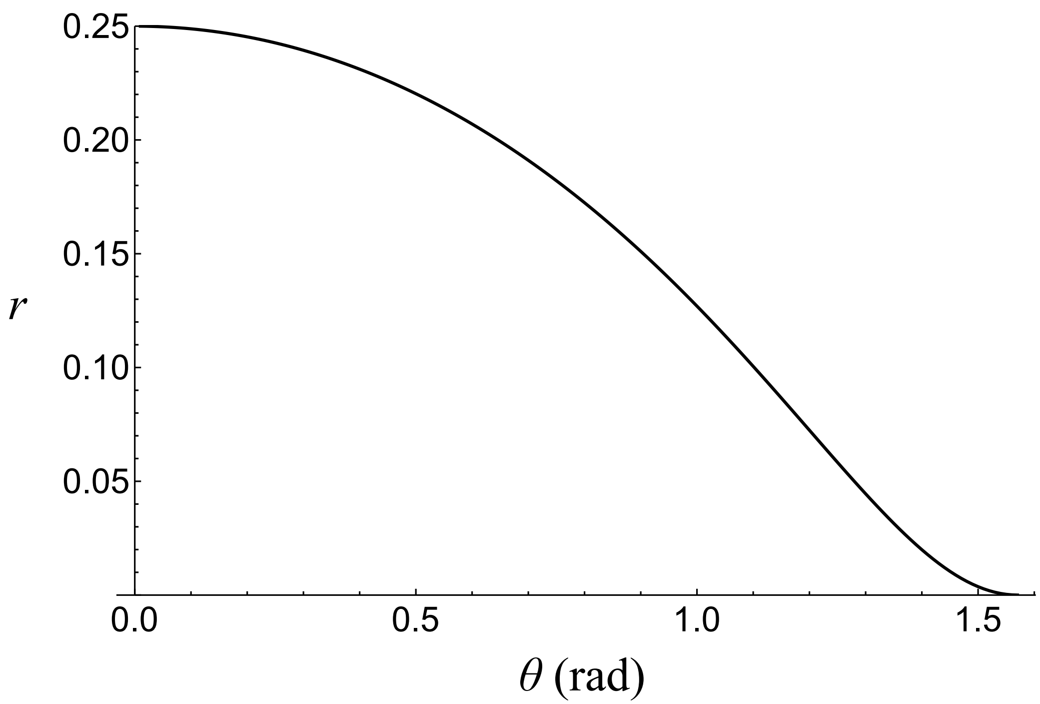
Linear Dispersive Case

$$h\zeta' + \frac{1}{4}\zeta + \frac{3}{2}\alpha_0\zeta\zeta' + \frac{1}{6}h^3\tan^2\theta\,\zeta''' = 0 \quad \text{becomes} \quad \boxed{\frac{1}{6}h^3\tan^2\theta\,\zeta''' + h\zeta' + \frac{1}{4}\zeta = 0}$$

This is a third order Euler-type equation. Let $\zeta = C_0 h^{-r}$.

Then, we find the following polynomial to satisfy the equation:

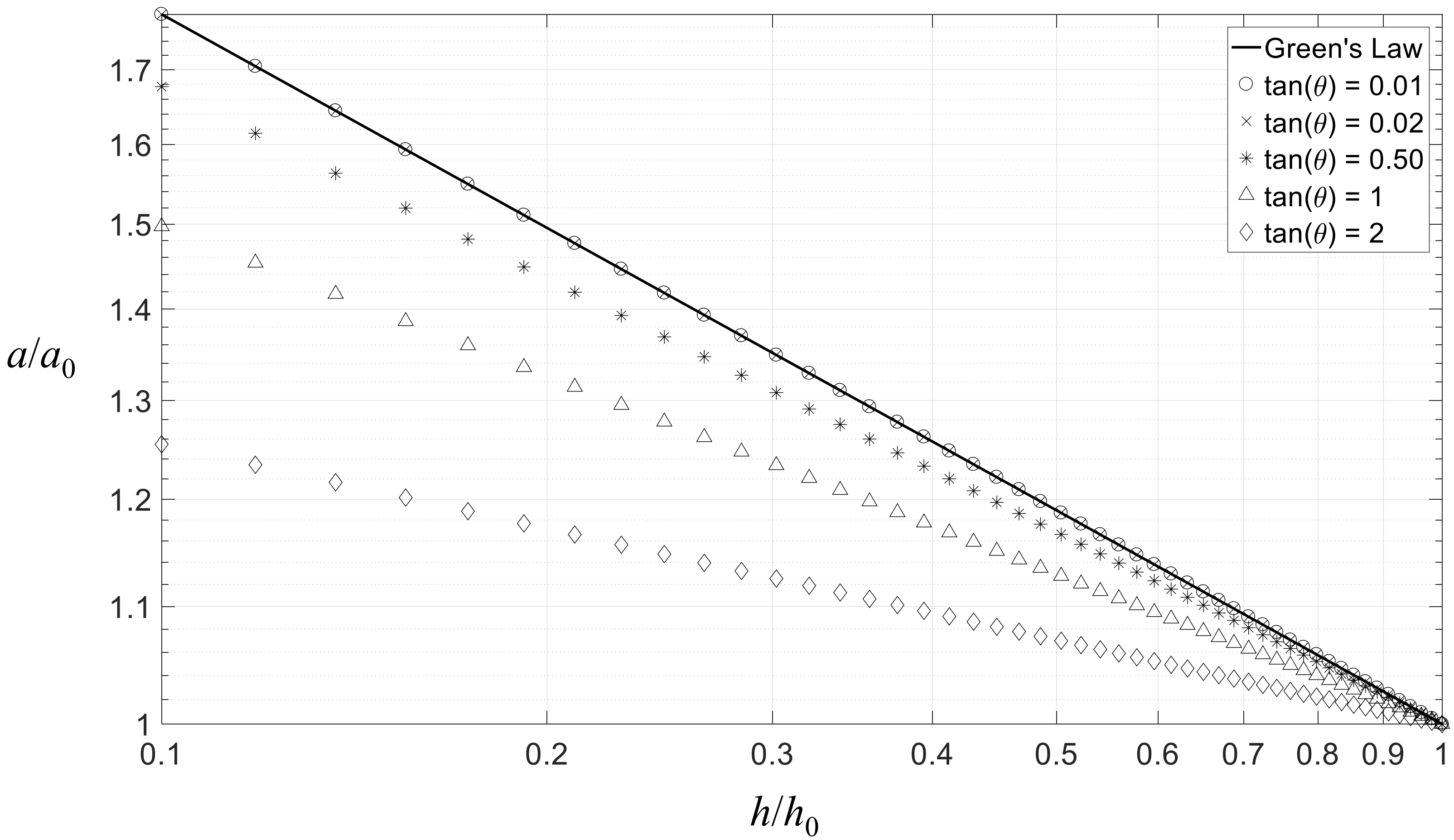
$$\tan^2\theta\left(\tfrac{1}{6}r^3 + \tfrac{1}{2}r^2 + \tfrac{1}{3}r\right) + r - \tfrac{1}{4} = 0$$



$r \sim$ ¼ for $\theta \ll 1$, i.e. Green's law.
$r \sim 0.1$ for $\theta \sim \pi/3$;
$r \sim 0$ for $\theta \sim \pi/2$.

$$a \propto h^{-r};\ r < \tfrac{1}{4}$$

Linear Dispersive Case



Very small dependency to the beach slope θ when it is ‘small’.

$$a \propto h^{-r};\ r < \tfrac{1}{4}$$

Numerical Approach

The Euler Code
Higher-Order Pseudo-Spectral Method

Dommermuth and Yue (1987); Tanaka (1993); Jia (2014)

$$\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0 \qquad \text{in } -h_0 + \tilde{\zeta}(\tilde{x},\tilde{t}) \le \tilde{z} \le \tilde{\eta}(\tilde{x},\tilde{t})$$

$$\tilde{\zeta}_{\tilde{t}} + \tilde{\zeta}_{\tilde{x}}\,\tilde{\phi}_{\tilde{x}} - \tilde{\phi}_{\tilde{z}} = 0 \qquad \text{on } \tilde{z} = -h_0 + \tilde{\zeta}(\tilde{x},\tilde{t})$$

$$\tilde{\eta}_{\tilde{t}} + \tilde{\phi}_{\tilde{x}}\,\tilde{\eta}_{\tilde{x}} - \tilde{\phi}_{\tilde{z}} = 0 \qquad \text{on } \tilde{z} = \tilde{\eta}(\tilde{x},\tilde{t})$$

$$\tilde{\phi}_{\tilde{t}} + g\,\tilde{\eta} + \tfrac{1}{2}\left(\tilde{\phi}_{\tilde{x}}^2 + \tilde{\phi}_{\tilde{z}}^2\right) = 0 \qquad \text{on } \tilde{z} = \tilde{\eta}(\tilde{x},\tilde{t})$$

$$(\tilde{x},\tilde{z}) = (\lambda_0 x,\, \lambda_0 z); \quad \tilde{t} = (\lambda_0 / c_0)\, t; \quad \tilde{\phi} = \lambda_0\, c_0\, \phi; \quad \tilde{\eta} = \lambda_0\, \eta; \quad \tilde{\zeta} = \lambda_0\, \zeta$$

The kinematic and dynamic boundary conditions at the free surface, $z = \eta\ (x,\ t)$:

$$\eta_t = (1+\eta_x^2)\phi_z - \phi_x^s\,\eta_x \quad \text{and} \quad \phi_t^s = \tfrac{1}{2}\left(1+\eta_x^2\right)\left(\phi_z\right)^2 - \tfrac{1}{2}\left(\phi_x^s\right)^2 - \eta$$

where $\phi^s(x,\, t) = \phi(x,\, \eta(x,t),\, t)$

The Euler Code: (Dommermuth and Yue,1987)

Taking a perturbation expansion of velocity potential ϕ together with the Taylor expansion about $z = 0$:

$$\phi^s(x,t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\eta^k}{k!} \frac{\partial^k \phi_m}{\partial z^k}\bigg|_{z=0}$$

Introduce a linear combination of basis functions which also satisfy Laplace's equation: $\phi_m(x,z,t) = A_m(x,z,t) + B_m(x,z,t)$

Modeling the vertical velocity:

$$A_m(x,z,t) = \sum_{n=0}^{\infty} A_{mn}(t) \frac{\cosh(k_n(z+h))}{\cosh k_n h} e^{i k_n x} \leftrightarrow \frac{\partial^k A_m}{\partial z^k} = 0 \text{ at } z = -h \text{ when } k \text{ is odd,}$$

$$B_m(x,z,t) = B_{m0}(t) z + \sum_{n=1}^{\infty} B_{mn}(t) \frac{\sinh k_n z}{\cosh k_n h} e^{i k_n x} \leftrightarrow \frac{\partial^k B_m}{\partial z^k} = 0 \text{ at } z = 0 \text{ when } k \text{ is even.}$$

Therefore:

$$\phi^s(x,t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\eta^k}{k!} \frac{\partial^k}{\partial z^k}\left(A_m + B_m\right)\bigg|_{z=0}$$

For given ϕ^s, we expand for A_{mn} and B_{mn}.

The Euler Code

To determine B_{mn}, we need the bottom boundary condition:

At the bottom surface, $z = -h + \zeta(x)$

$$\phi(x, z = -h + \zeta, t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\zeta^k}{k!} \frac{\partial^k}{\partial z^k} (A_m + B_m) \Big|_{z=-h}$$

Substitute them to the bottom boundary condition at $z = -h + \zeta(x)$:

$$\zeta_x \phi_x - \phi_z = 0, \quad \text{on } z = -h + \zeta(x)$$

becomes

$$\zeta_x \phi_x - \sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\zeta^k}{k!} \frac{\partial^{k+1}}{\partial z^{k+1}} (A_m + B_m) \Big|_{z=-h} = 0, \quad \text{on } z = -h + \zeta(x)$$

For given ζ_x and ϕ^s, we successively determine B_{mn} and A_{mn} with the use of FFT.

Then, we express $\partial_z \phi(x, \eta, t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\eta^k}{k!} \frac{\partial^{k+1}(A_m + B_m)}{\partial z^{k+1}} \Big|_{z=0}$

The Euler Code

Substituting into the kinematic and dynamic boundary conditions at $z = 0$,

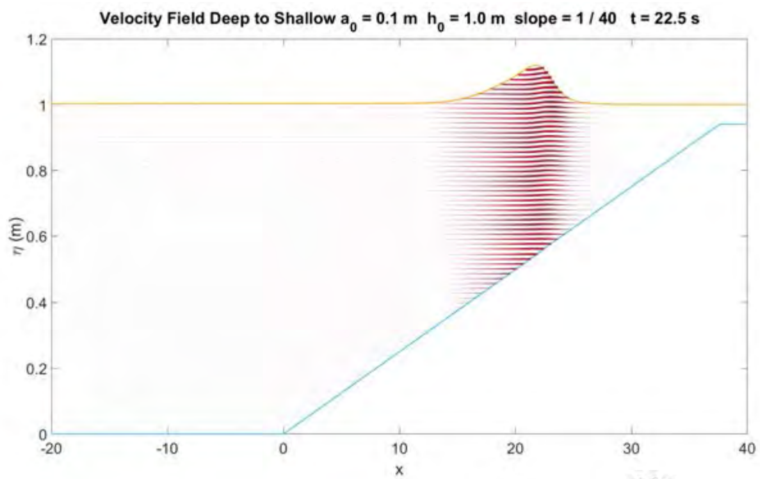
$$\begin{cases} \phi_t^s = \frac{1}{2}\left(1+\eta_x^2\right)\left(\sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\eta^k}{k!} \frac{\partial^{k+1}}{\partial z^{k+1}} \left(A_m + B_m\right)\Big|_{z=0} \right)^2 - \frac{1}{2}\left(\phi_x^s\right)^2 - \eta \\ \eta_t = (1+\eta_x^2) \sum_{m=1}^{M} \sum_{k=0}^{M-m} \varepsilon^m \frac{\eta^k}{k!} \frac{\partial^{k+1}}{\partial z^{k+1}} \left(A_m + B_m\right)\Big|_{z=0} - \phi_x^s \eta_x \end{cases}$$

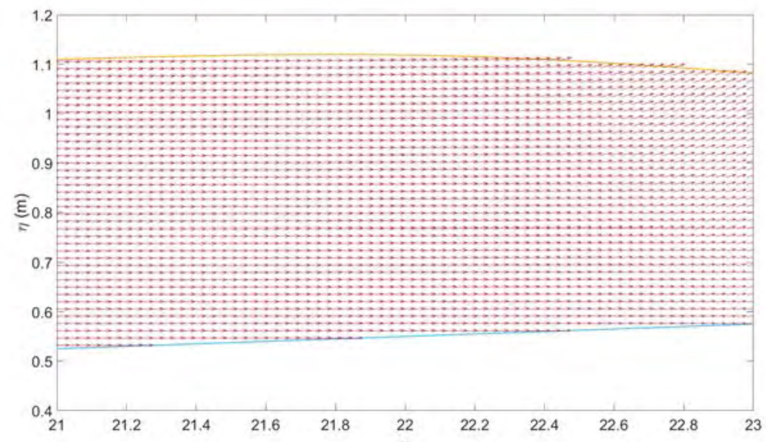
The above equations are no longer function of z, and we solve with the spectral method.

- Horizontal spatial derivatives in wavenumber space.
- We use $M = 5$ based on our sensitivity analysis for satisfying the no-flux boundary condition on the sloping bed .
- The 4th order Runge-Kutta method for time stepping.

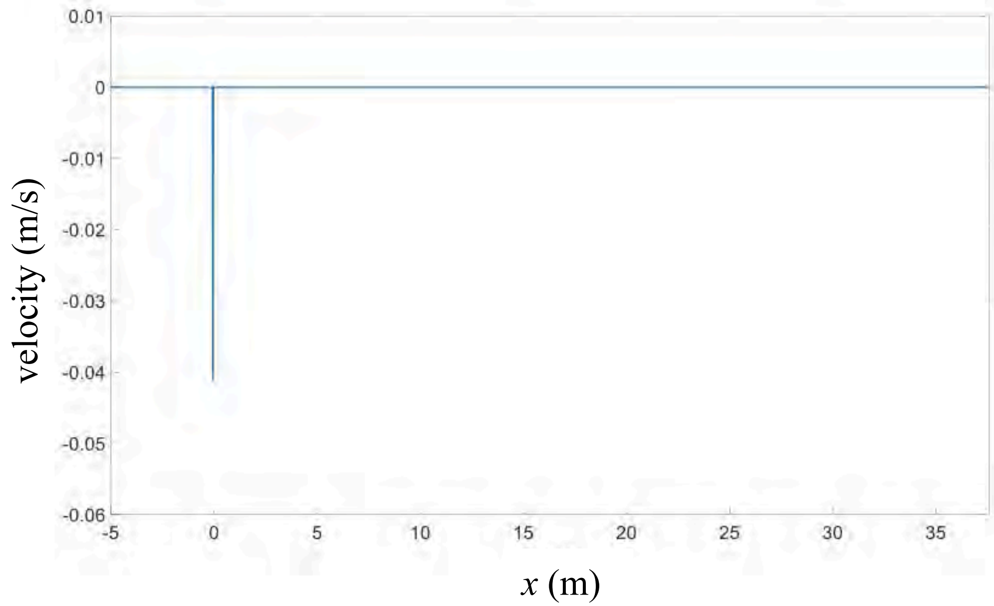
Validation

Our treatment of the bottom boundary condition is adapted from Dommermuth and Yue (1987), although they did not demonstrate the scheme. Hence, we validate it by numerically observing the no-flux condition on the sloping bed.



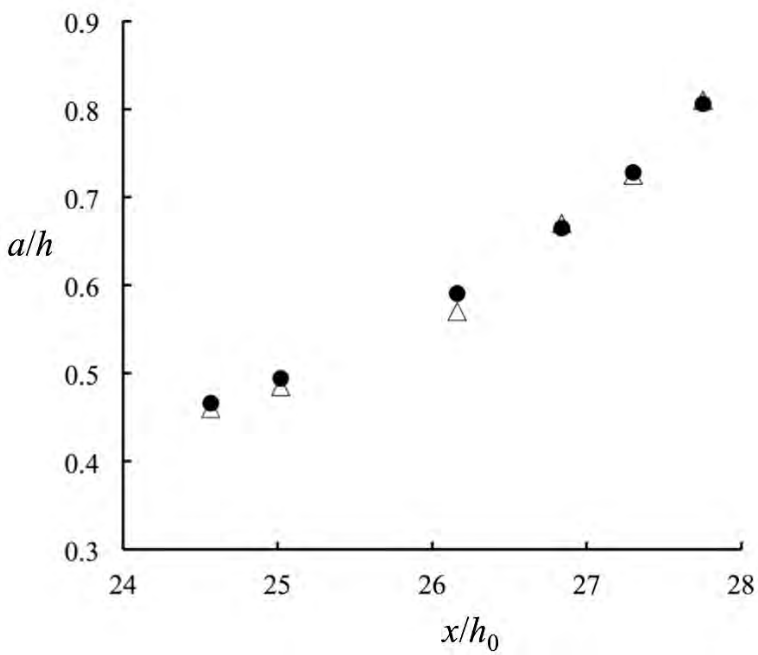


Velocity normal to the bed boundary vanishes except at $x = 0$ (beach toe).



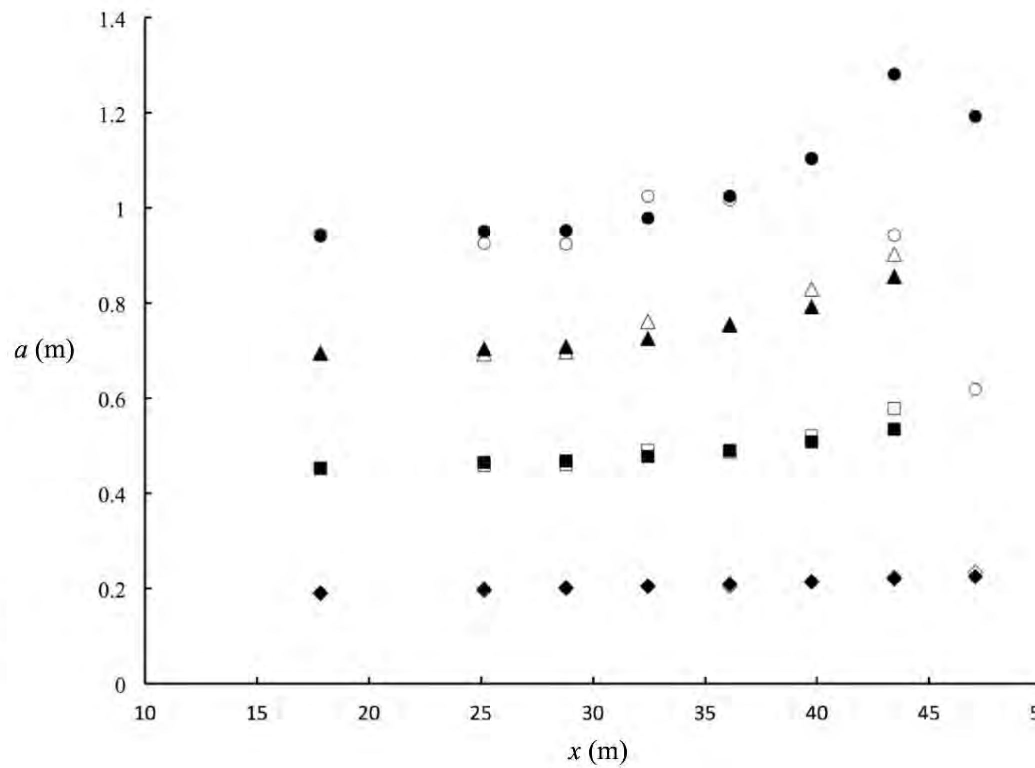
Validations of the Numerical Results

Comparison of numerically predicted shoaling (solid circles) with the laboratory data by Grilli (1994)



tan θ =1/34.7; a_0 = 0.044m, h_0 = 0.44m.

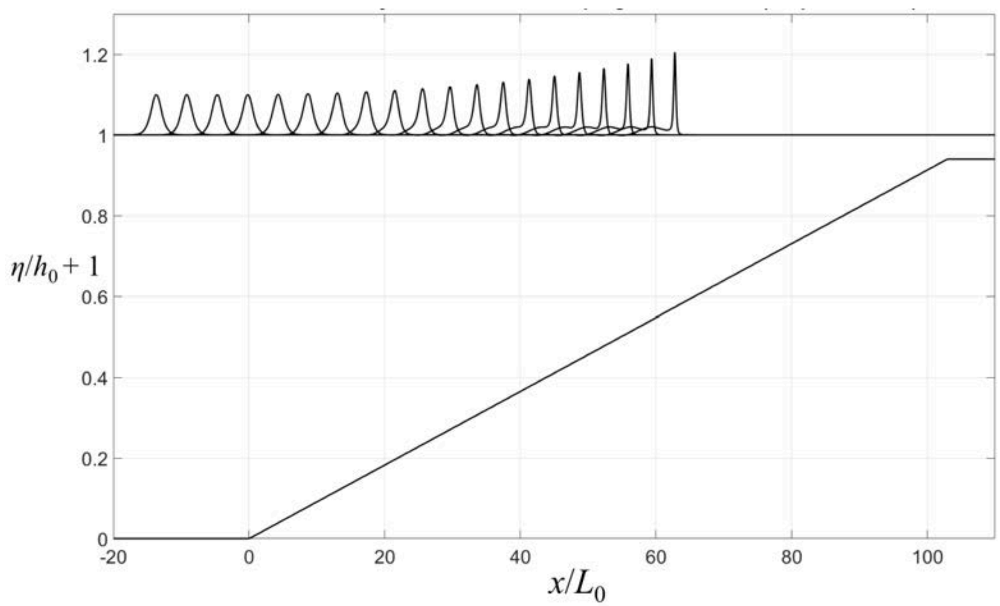
Comparisons of numerical values (solid marks) with the large-scale laboratory experiments (hollow marks)



Composite beach: tan θ_1 =1/12 and tan θ_2 =1/24; h_0 = 1.888m. circles: a_0 = 1.038 m, triangles: a_0 = 0.755 m, squares: a_0 = 0.472 m, diamonds: a_0 = 0.189 m.

Results

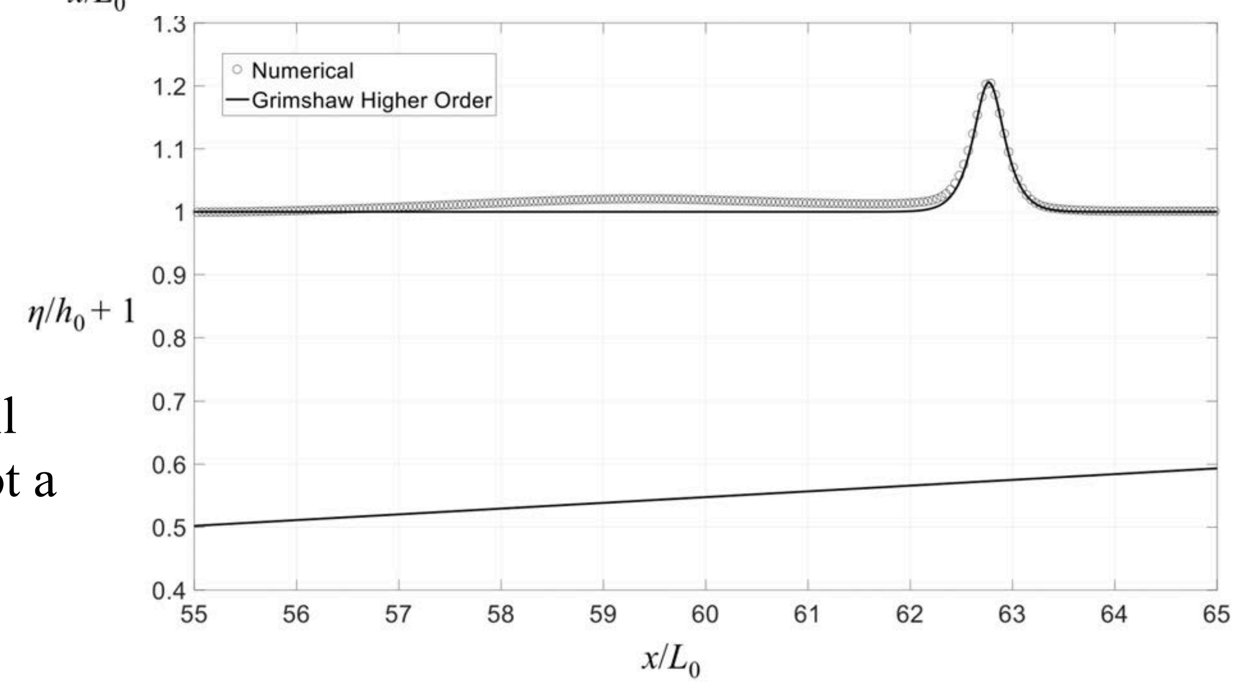
Shoaling on a Very Mild Slope: tan θ = 1/400



$\alpha_0 = 0.1$

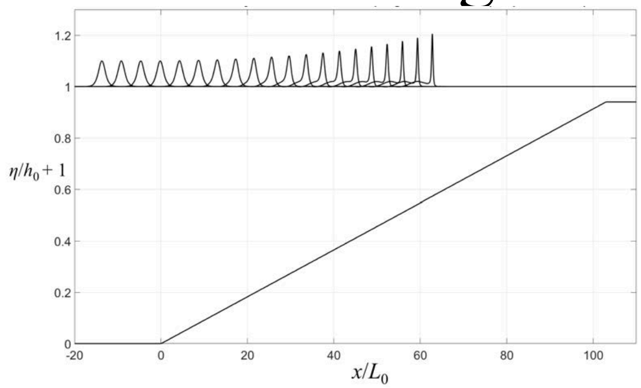
Adiabatic evolution?

$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}} = 0.009$$



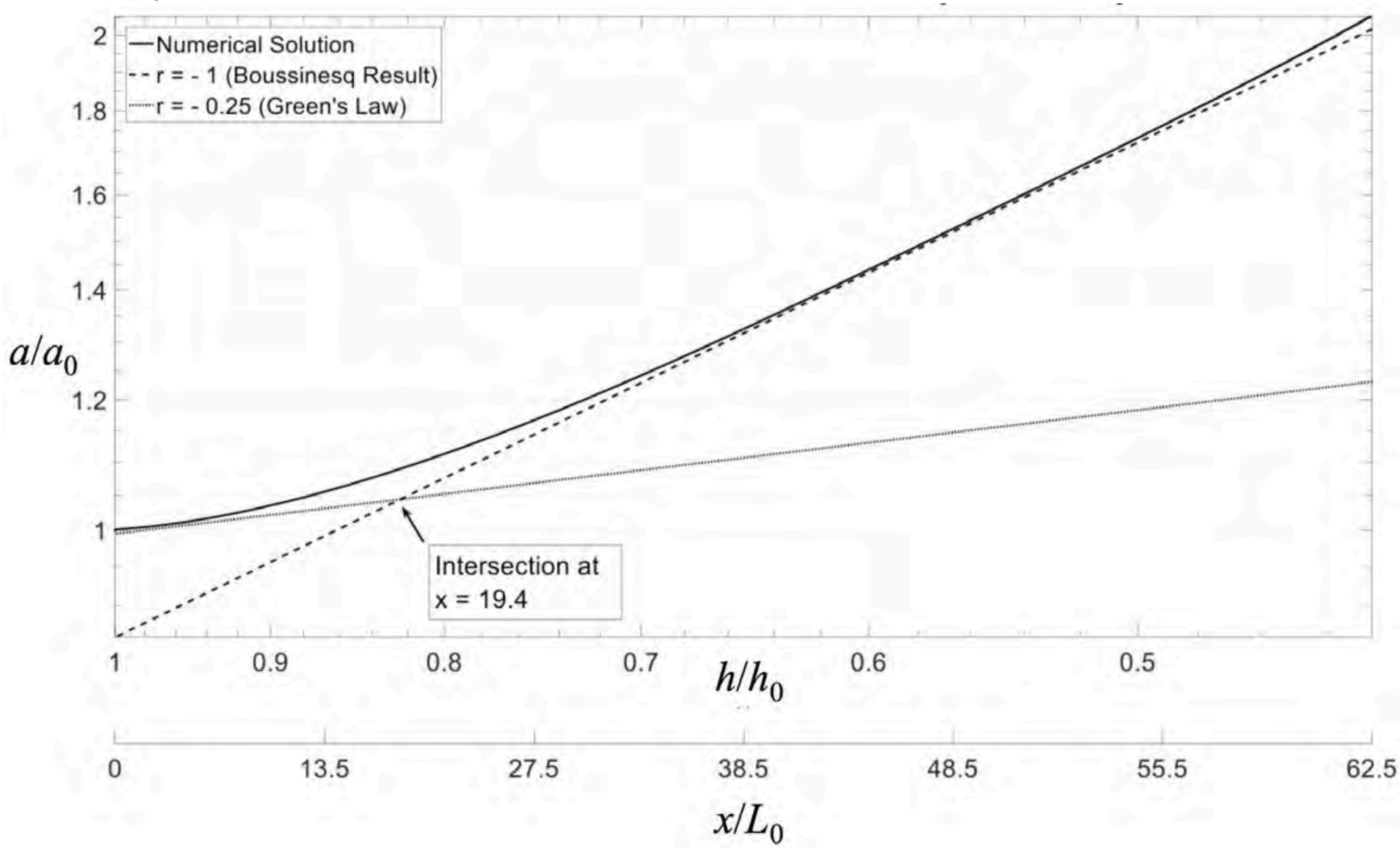
Note the trailing tail formation that is not a soliton.

Shoaling on a Very Mild Slope: tan θ = 1/400

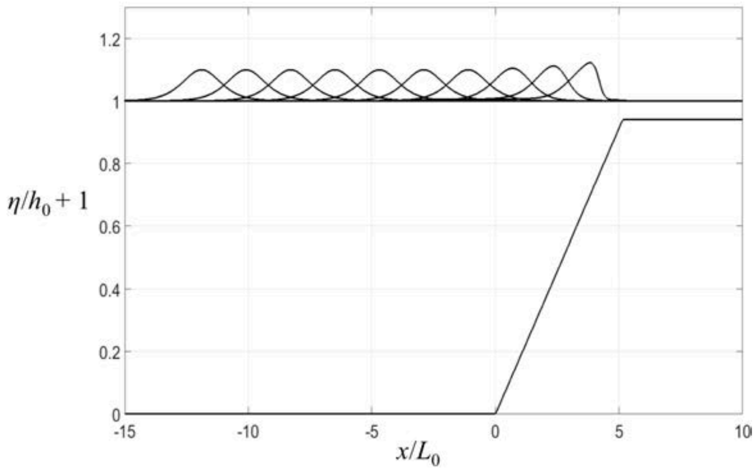


$$\alpha_0 = 0.1 \qquad \gamma \equiv \frac{L_0}{L_b} = 0.009$$

Adiabatic evolution process

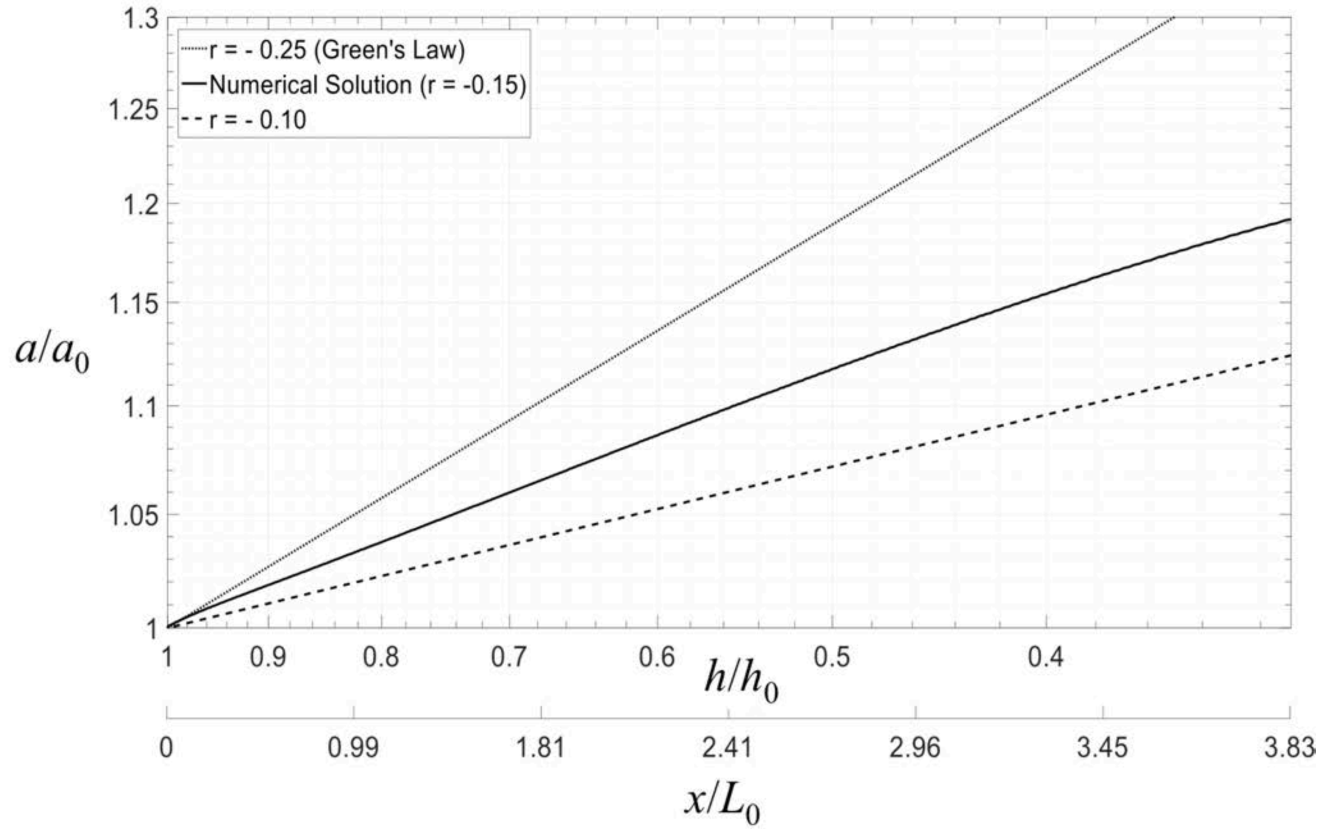


Shoaling of Wave for tan θ = 1/20



$$\alpha_0 = 0.1$$

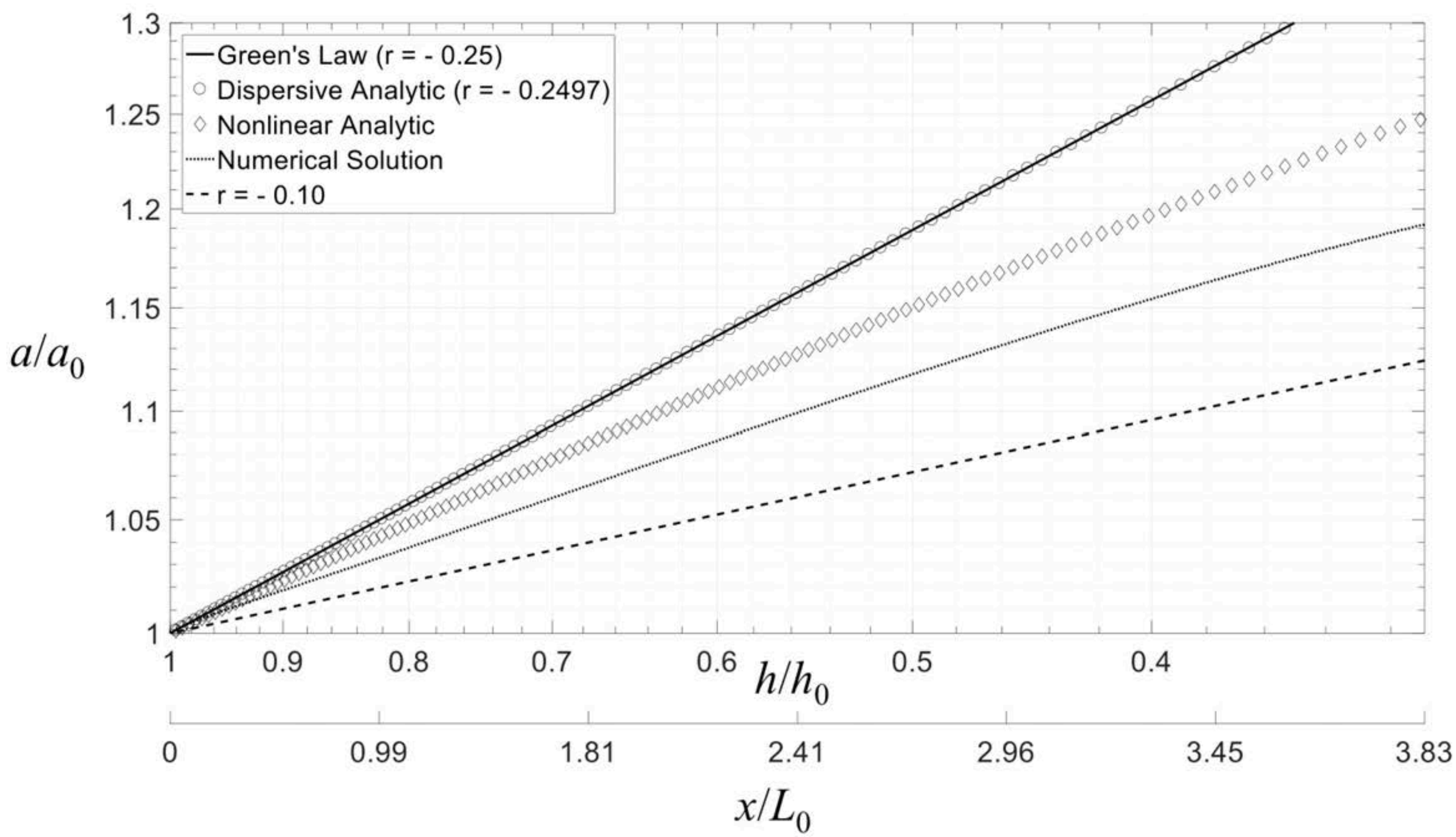
$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}} = 0.18$$



Shoaling of Wave for tan θ = 1/20

$$\alpha_0 = 0.1; \quad \gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}} = 0.18$$

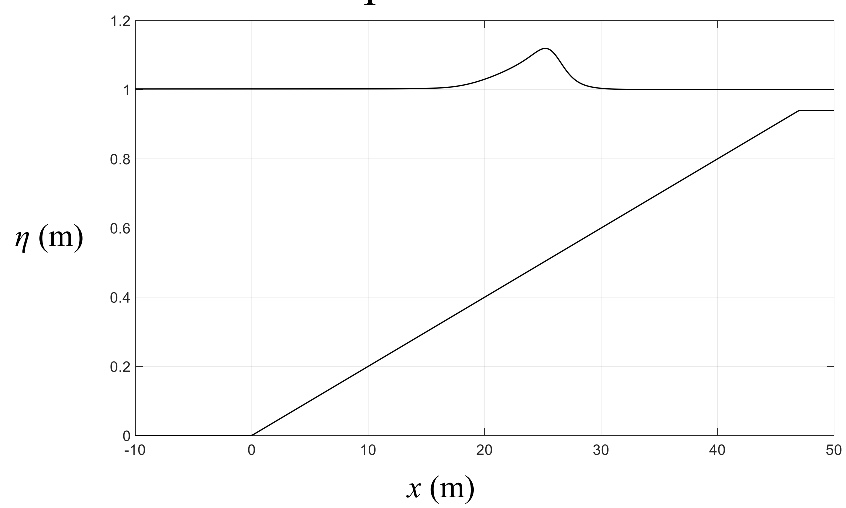
Comparison with the results of the vKdV theory.



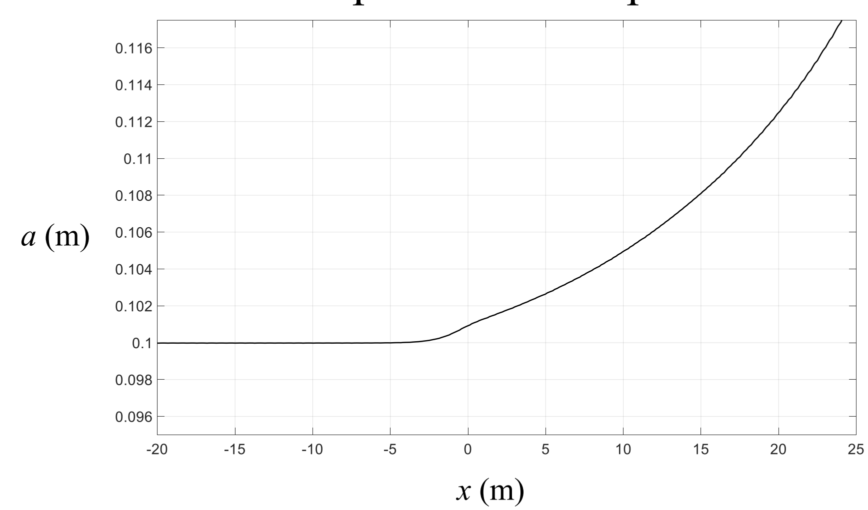
Momentum Flux and Wave Reflection

$a_0 = 0.1\text{m}$; $h_0 = 1.0$ m; $\tan\theta = 0.02$

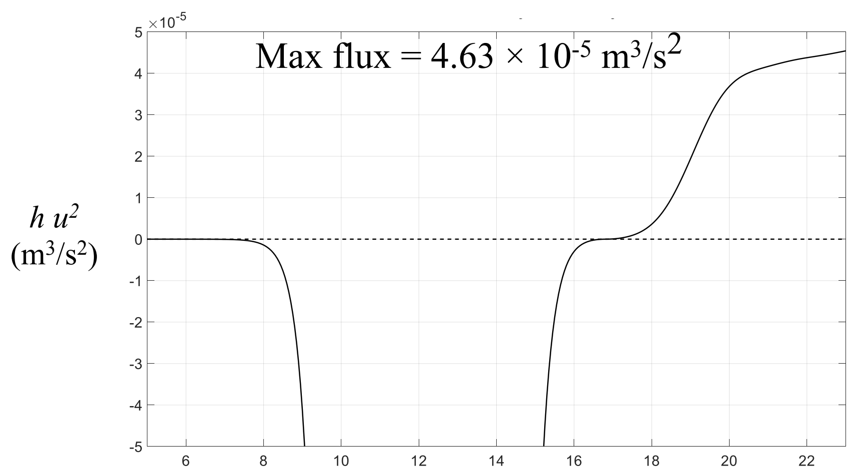
Wave profile at $t = 23.6$ s



Amplification in space

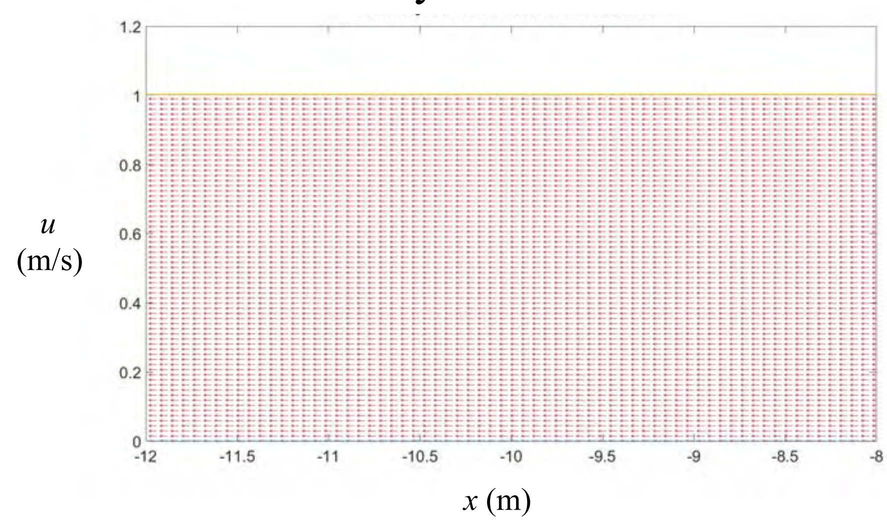


Momentum flux at $x = -10$m



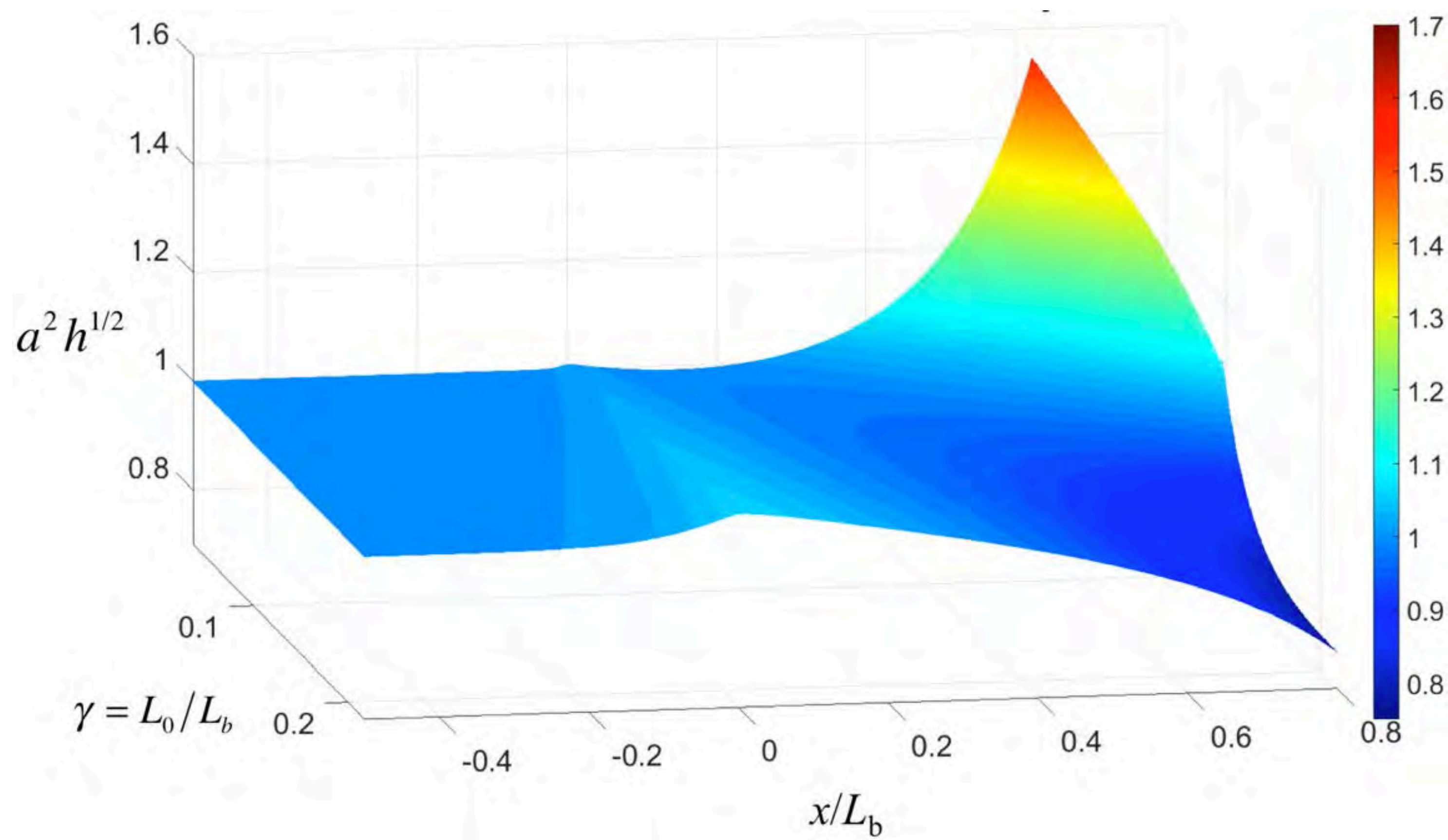
Max flux = $0.1146\ \text{m}^3/\text{s}^2$

Velocity field at $t = 23.6$ s



Evolution of Energy Flux: $\alpha_0 = 0.10$

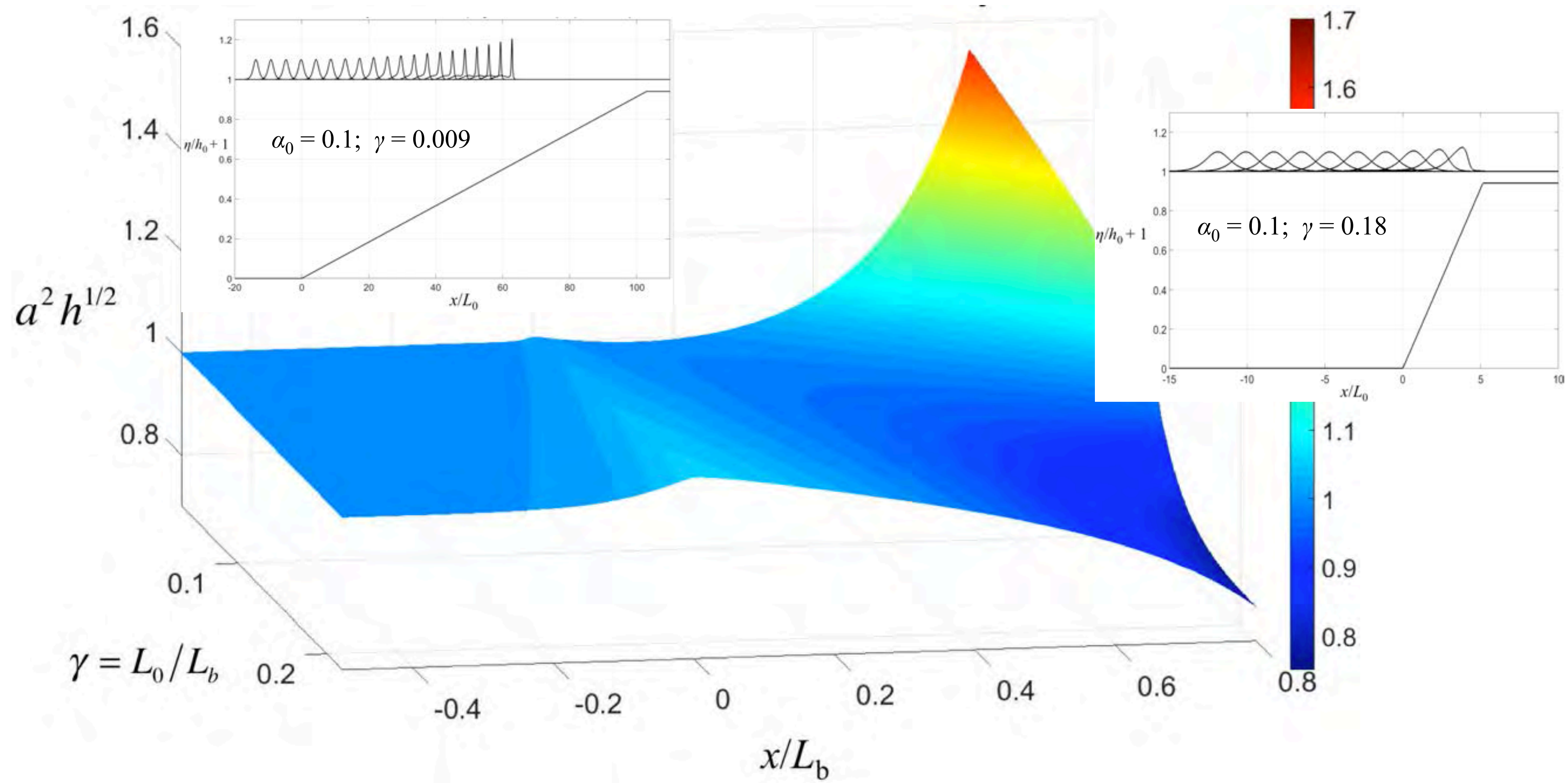
$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4h_0}{3a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$



Amplifies toward the beach toe. The amplification growth is slower than Green's law, when γ is large. When γ is small, the growth rate can exceed that of Green's law near the shore: approach to the adiabatic evolution process but it breaks early.

Evolution of Energy Flux: $\alpha_0 = 0.10$

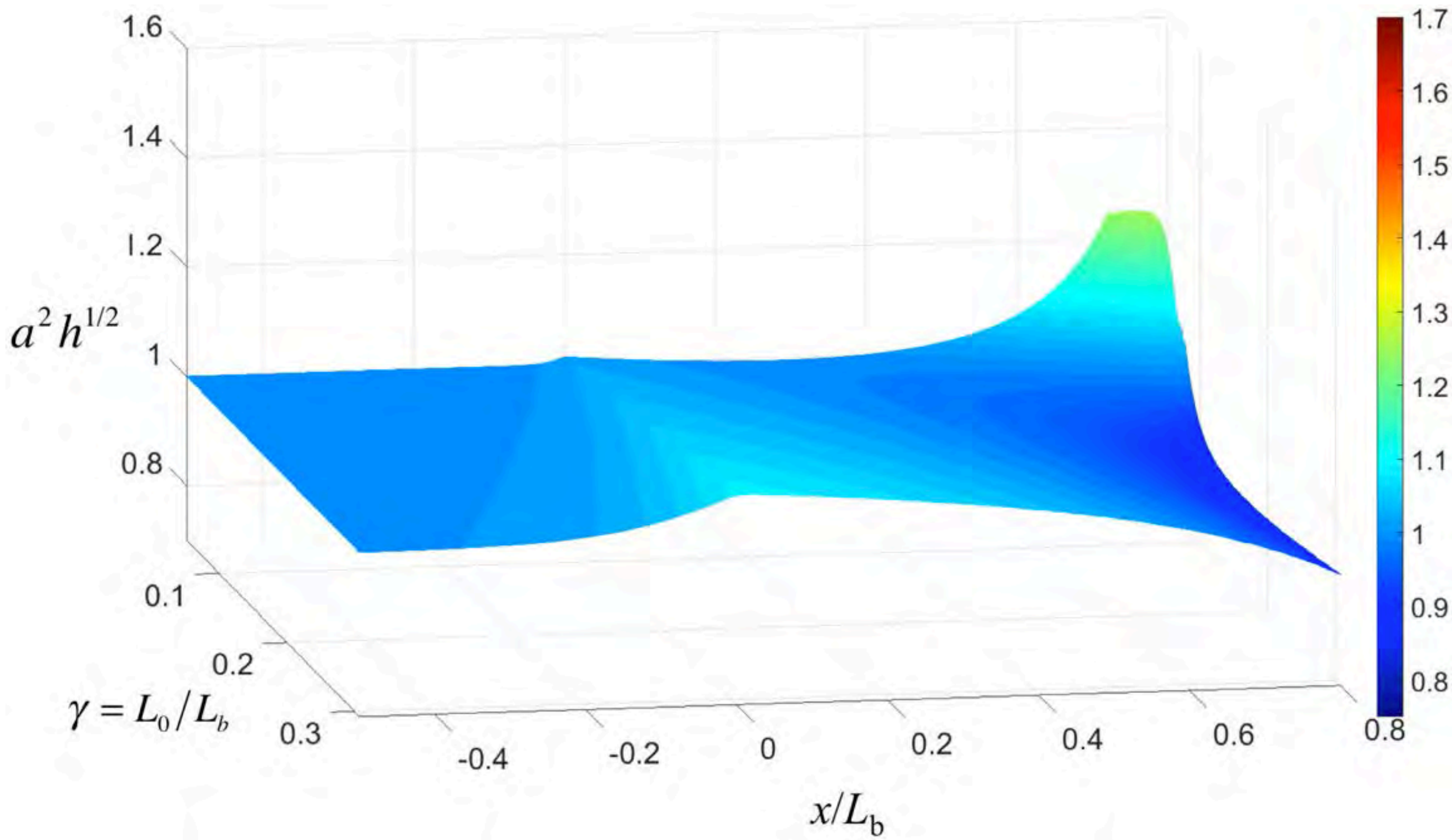
$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4h_0}{3a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$



Amplifies toward the beach toe. The amplification growth is slower than Green's law, when γ is large. When γ is small, the growth rate can exceed that of Green's law near the shore: approach to the adiabatic evolution process but it breaks early.

Evolution of Energy Flux: $\alpha_0 = 0.05$

$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4h_0}{3a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$



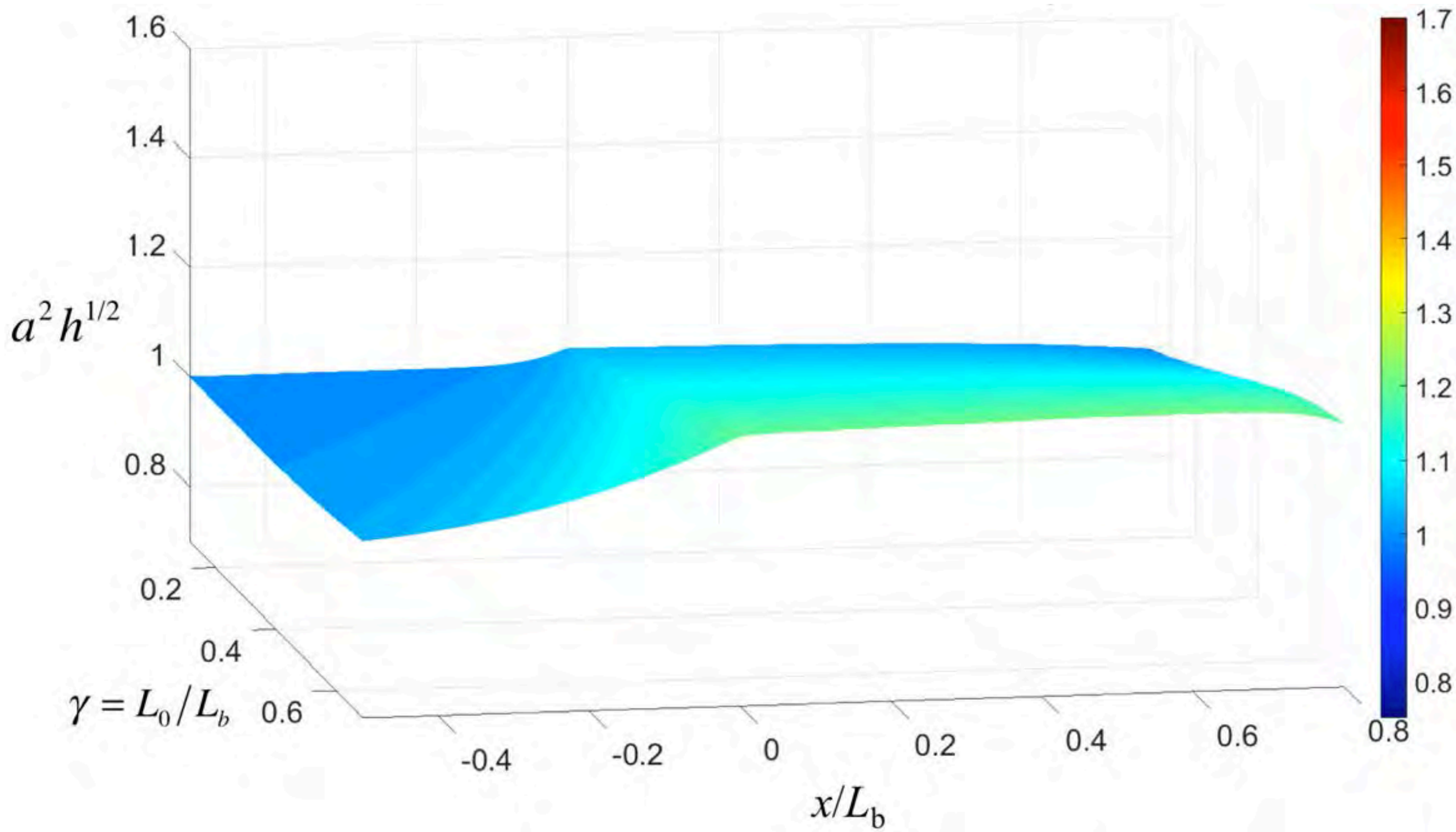
Amplifies toward the beach toe.

The amplification growth is slower than Green's law.

When γ is small, the growth rate can exceed that of Green's law near the shore.

Evolution of Energy Flux: $\alpha_0 = 0.01$

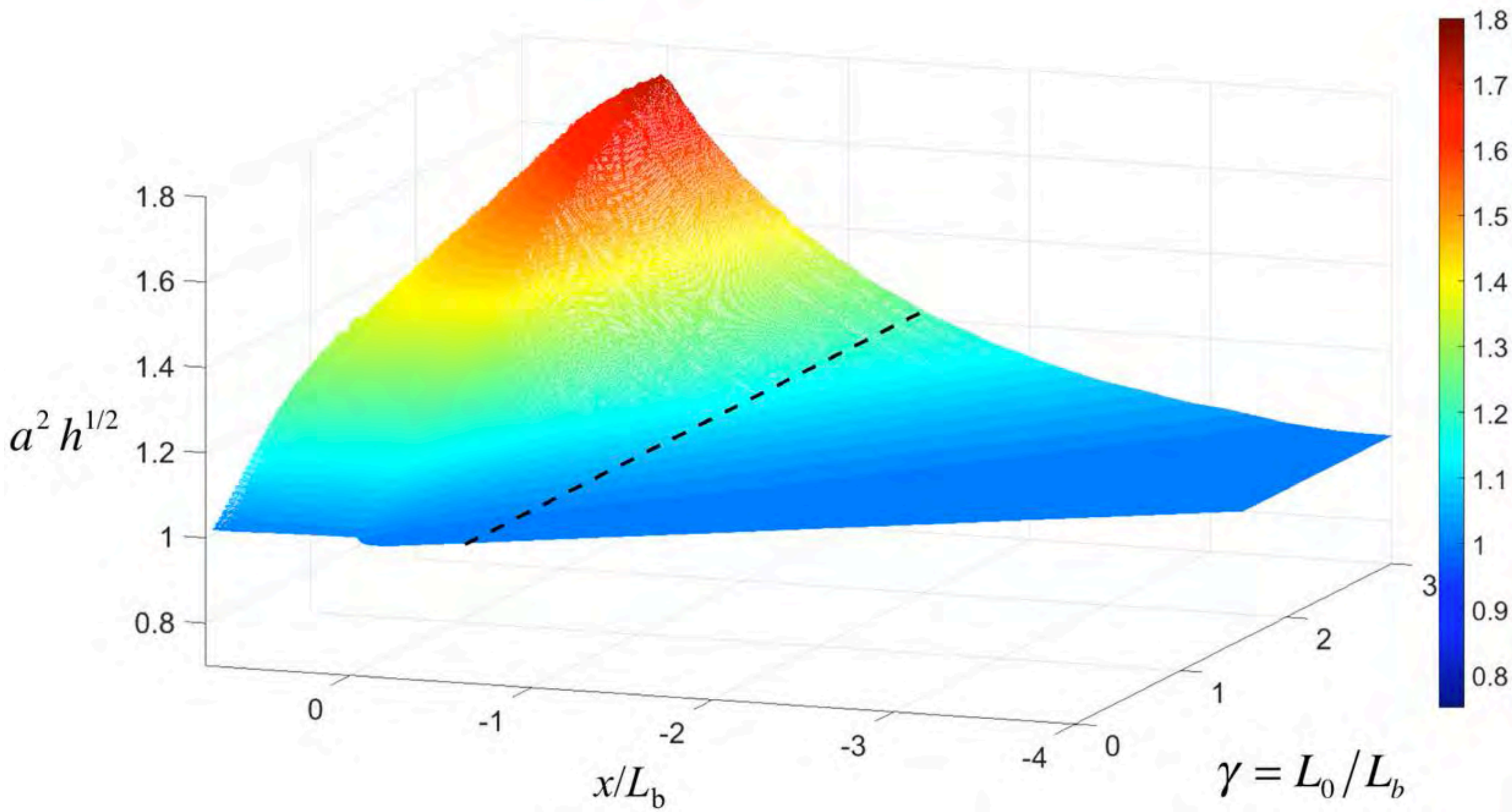
$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4h_0}{3a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$



Amplifies toward the beach toe, but approximately follows Green's law thereafter: $a^2h^{1/2} \sim$ constant. The steeper the beach slope, the greater the amplification prior to reaching the beach toe.

Evolution of Energy Flux: $\alpha_0 = 0.003$

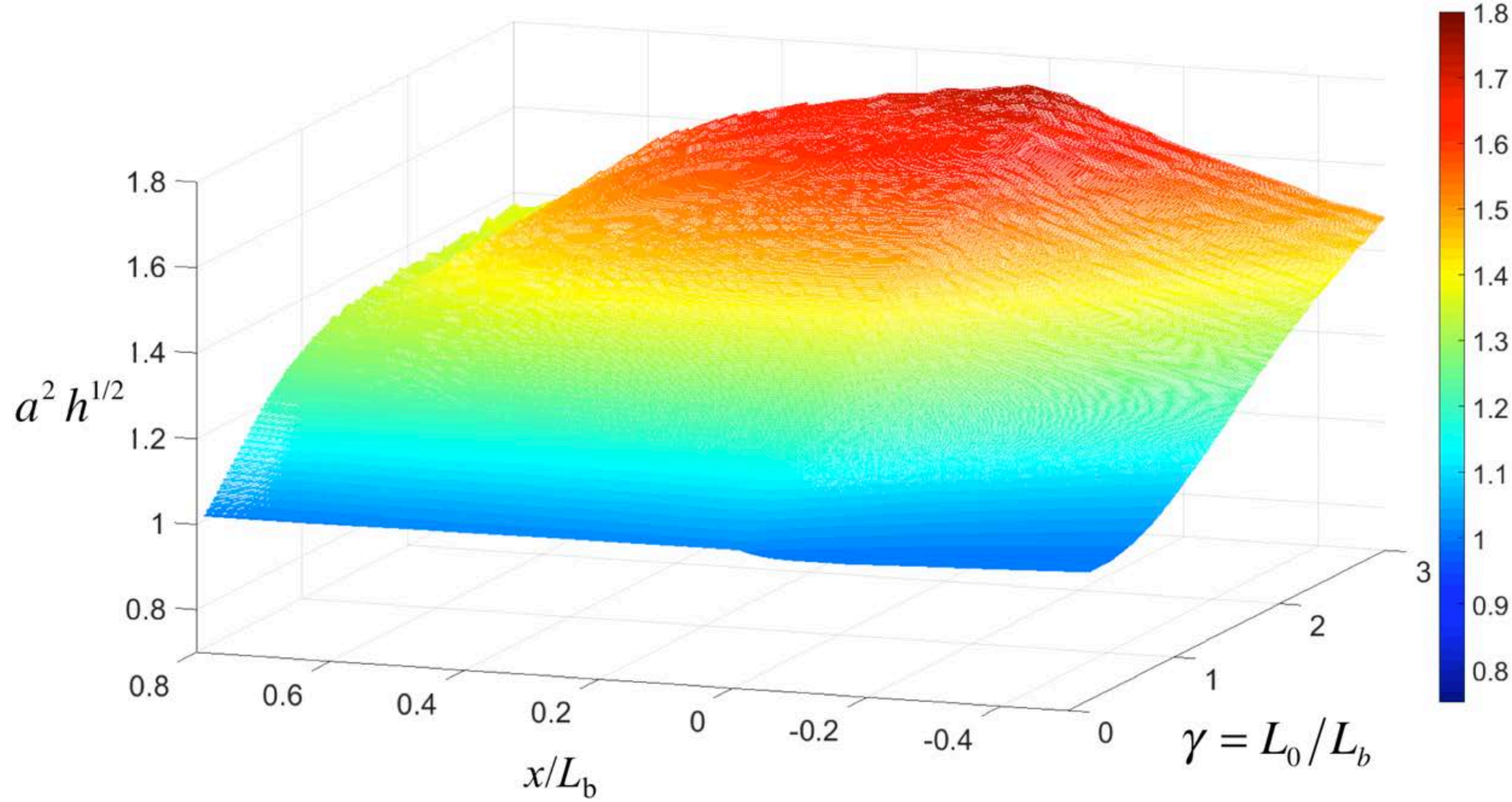
$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4\,h_0}{3\,a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$



See the wave amplification starts far offshore due to reflection from the steep beach when γ is large. This because the wavelength is so long in comparison with the (steep) beach length.

Evolution of Energy Flux: $\alpha_0 = 0.003$

$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4h_0}{3a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$

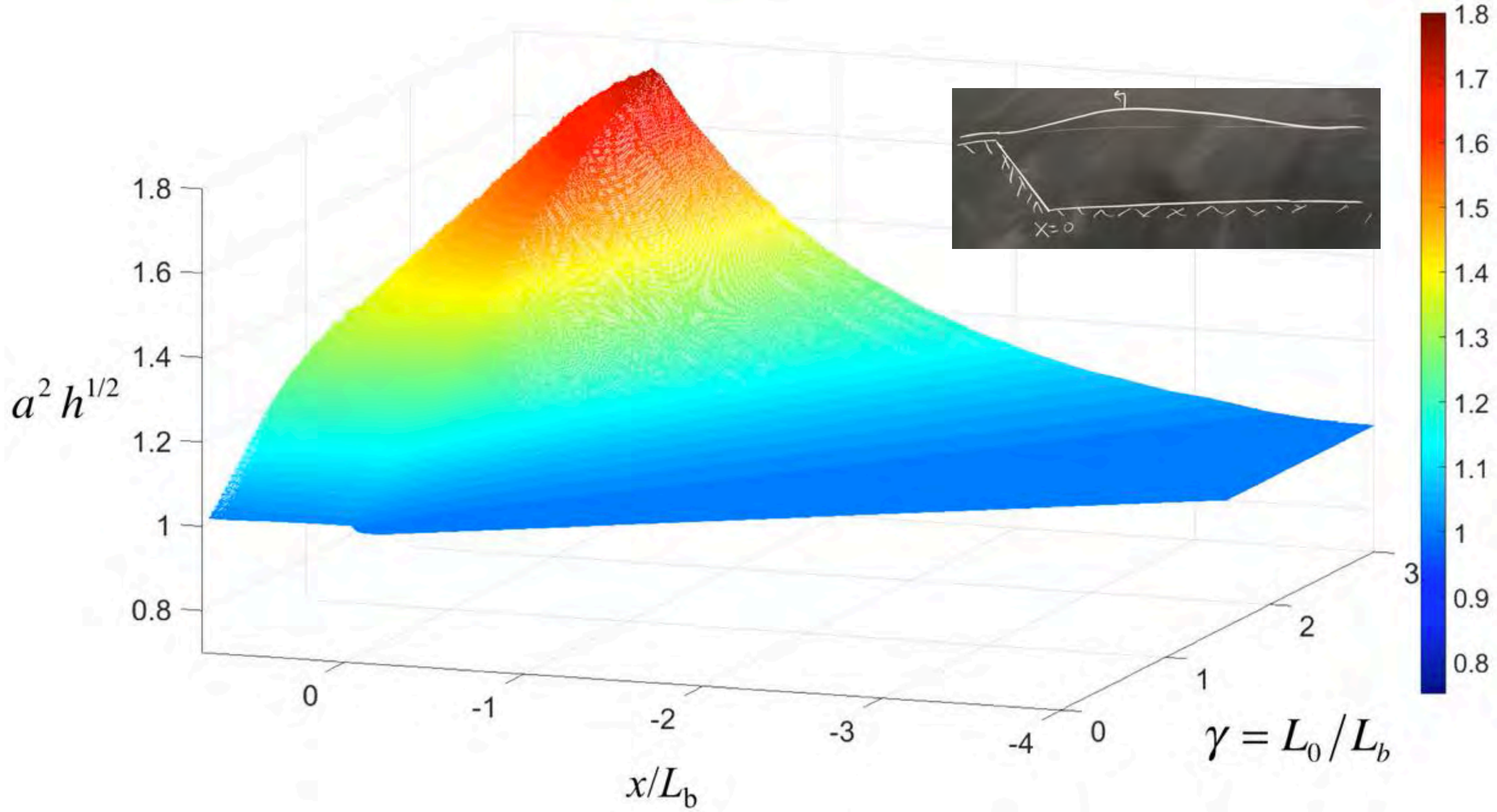


Follows Green's law when γ is small, say $\gamma < 1.0$.

Then, the amplification becomes $a \propto h^{-r}$; $r < \frac{1}{4}$ for $\gamma > 1.0$.

Evolution of Energy Flux: $\alpha_0 = 0.003$

$$\gamma \equiv \frac{L_0}{L_b} = \sqrt{\frac{4\tan^2\theta}{3\alpha_0}}; \quad L_0 = \sqrt{\frac{4h_0}{3a_0}}\,h_0; \quad a^2 h^{1/2} \leftarrow \frac{a^2 h^{1/2}}{a_0^2 h_0^{1/2}}$$



See the wave amplification starts far offshore due to reflection from the steep beach when γ is large. This because the wavelength is so long in comparison with the (steep) beach length.

Conclusion

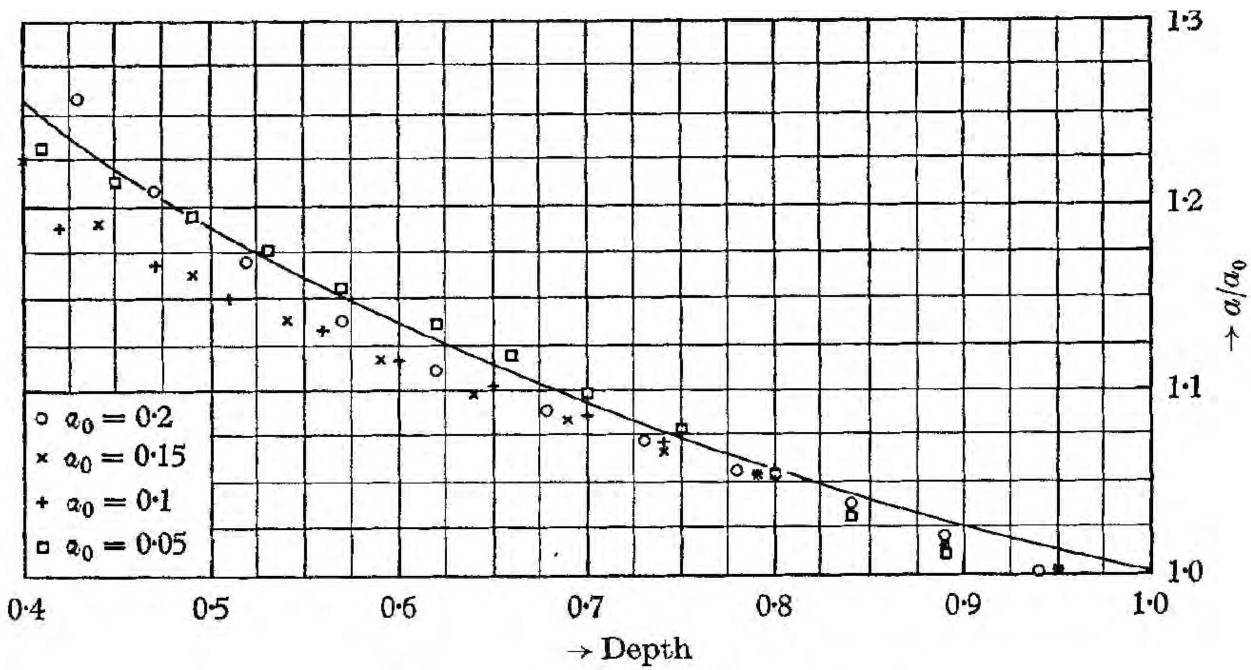
Summary

- Solitary wave amplifies while it approaches the beach toe.
- Shoaling of a solitary wave can follow Green's law ($a \propto h^{-1/4}$), when the nonlinearity parameter α_0 is small (< 0.01) and the beach slope parameter $\gamma = L_0/L_b$ is smaller than $O(1)$.
- Shoaling of a solitary wave can follow the adiabatic evolution ($a \propto h^{-1}$), when the nonlinearity parameter α_0 is large (~ 0.1) and the beach slope parameter $\gamma = L_0/L_b$ is very small ($< O(0.01)$).
- Shoaling of a solitary wave takes place at the slower rate ($a \propto h^{-r}$; $r < 1/4$) than Green's law, when the nonlinearity parameter $\alpha_0 \sim O(0.1)$ and the beach slope parameter $\gamma = L_0/L_b$ is also small ($\sim O(0.1)$). This is the vKdV limit.
- The findings are qualitatively consistent with the numerical results provided by Peregrine (1967).

Peregrine (1967)

$$\begin{cases} \bar{u}_t + \bar{u}\,\bar{u}_x + \eta_x = \tfrac{1}{3}\theta^2 x^2 \bar{u}_{xxt} + \theta^2 x \bar{u}_{xt}, \\ \eta_t + \left[(\theta x + \eta)\bar{u}\right]_x = 0. \end{cases}$$

Extension of the Boussinesq equation:
x points offshore from the initial shoreline



Numerical results of the solitary-wave shoaling: $\theta = 1/20$. The solid line represents Green's law. Also note the different rate of amplification with α_0.

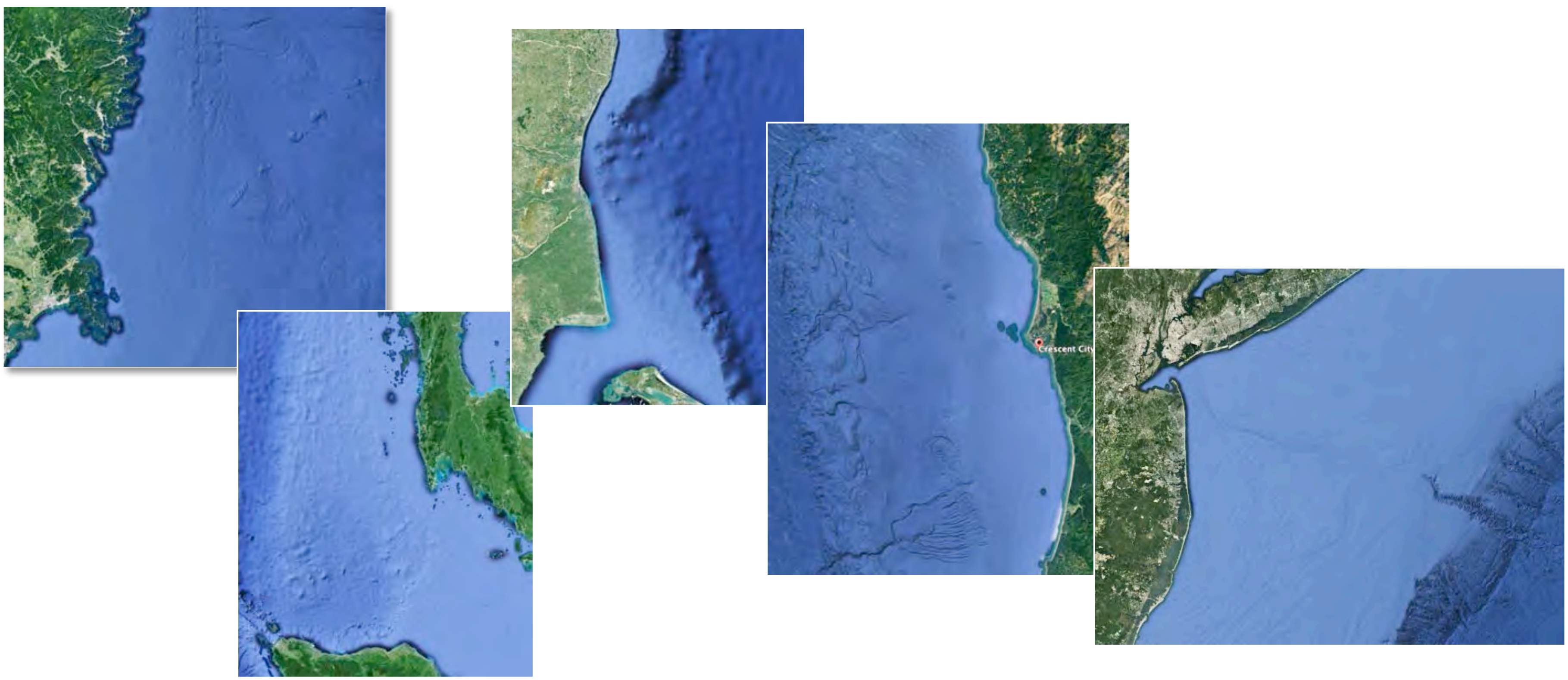
$\alpha_0 = 0.05$, $r \approx ¼$; $\alpha_0 = 0.1$ & 0.15, $r < ¼$; $\alpha_0 = 0.2$, $r > ¼$.

Summary

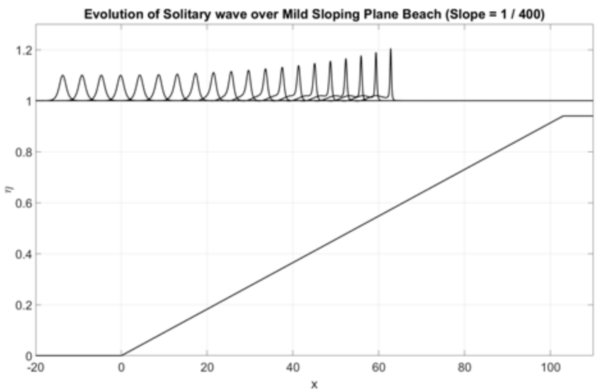
- There must be a factor(s) other than the parameter $\gamma = L_0/L_b$ that influence the shoaling. Possibly,
    1. Wave reflection at the beach toe.
    2. Wave runup onto the dry shore.
    3. Development of the wave skewness.
- For real co-seismic tsunamis, $\alpha_0 = O(10^{-3})$ and $\gamma = O(10^{-1}) \sim O(1)$, the wave should shoal as the rate less than Green's law. $r \leq 1/4$.
- To realize the adiabatic evolution ($r = 1$), $\gamma < O(10^{-2})$ and $\alpha_0 \geq O(10^{-1})$. It is possible to happen for a landslide generated tsunami. But it would likely radiate out because of a small source area for a landslide.

Shoaling of Tsunamis

Event	$\alpha_0 = a_0/h_0$	$\tan\theta$	$\Upsilon = L_0/L_b$	
2011 Heisei East Japan	0.003	0.02	0.42	
2004 Indian Ocean, Thailand	0.003	0.003	0.063	
2004 Indian Ocean, India	0.0007	0.03	1.1	
Crescent City, California	0.0005	0.045	2.3	Tsunami Source in Alaska
Off New York	0.1	0.0025	0.0091	Submarine Landslide

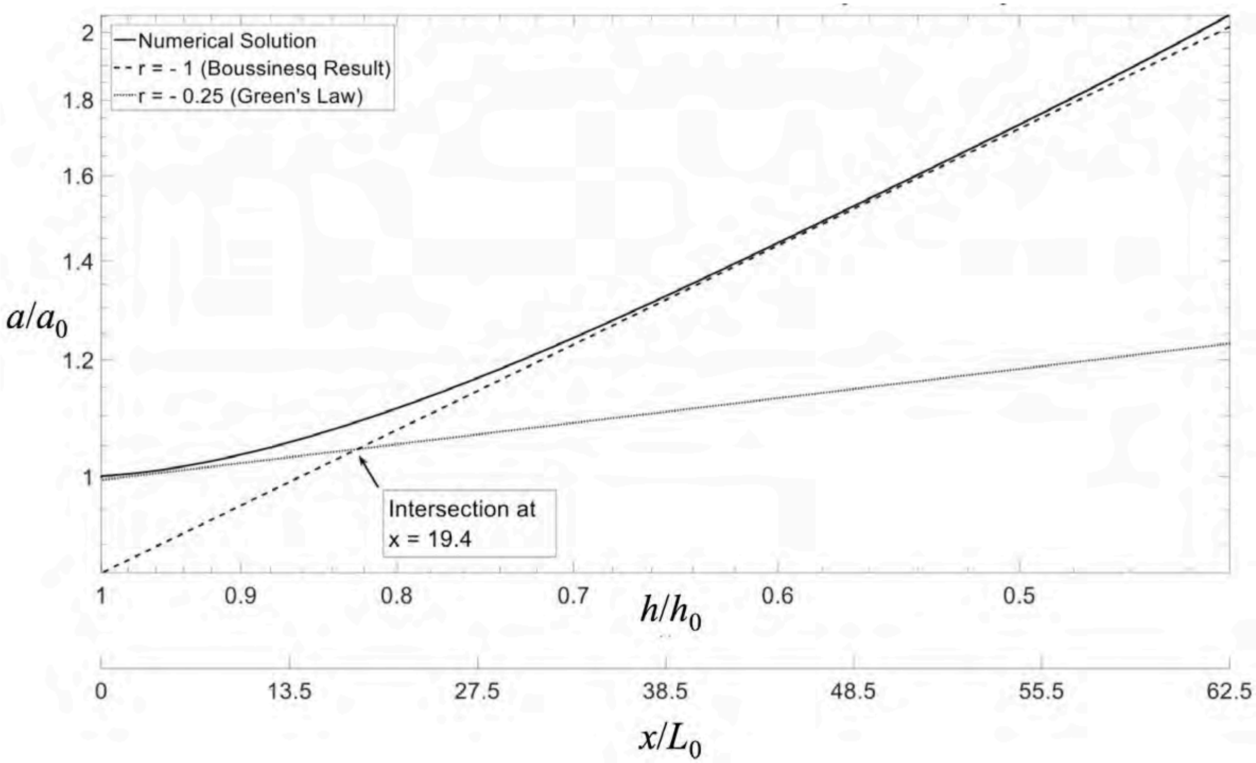


Shoaling on a Very Mild Slope: tan θ = 1/400



$$\alpha_0 = 0.1 \qquad \gamma \equiv \frac{L_0}{L_b} = 0.009$$

For example, h_0 = 200 m, a_0 = 20 m !!
Possible offshore landslide generated tsunamis off New York. But it would likely radiate out because of a small source area.



Adiabatic shoaling process unlikely occurs for tsunamis.